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On the existence, uniqueness and regularity of strong solutions to a stochastic 2D Allen-Cahn-Magnetohydrodynamic model

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Abstract

A bounded domain of $\mathbb{R}^2$ is used to investigate a stochastic coupled model of the Allen-Cahn equations and the stochastic magnetohydrodynamic equations. The model consists of the Allen-Cahn equation, Navier-Stokes equations and Maxwell equations (magnetic field) which are nonlinearly coupled through convection, stresses and Lorentz forces and expatiates the flow of the mixture of two incompressible, immiscible fluids under the influence of an electromagnetic field with stochastic perturbations. We demonstrate the existence, uniqueness and regularity of a strong solution. The method used is based on the Galerkin approximation, the stopping time technique and some feeble convergence principles in functional analysis.

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35R60, 60H15, 76W05.

Keywords

Stochastic, Magnetohydrodynamic, Allen-Cahn, Strong solution, Galerkin approximation.

1 Introduction

An important number of physical phenomena are elaborated by nonlinear partial differential equations. The Navier-Stokes (NS) equations are an illustration of this and are meant to describe the evolution of an incompressible fluid and variants of these equations are found, among others in meteorology, oceanography (see [5, 9, 10, 11, 15]). The Navier-Stokes equations coupled with the Maxwell equations of electromagnetism through the Lorentz force gives the magnetohydrodynamic (MHD) model which applies to plasma in particular and to many other fields in applied science.

Some years ago, the deterministic MHD system was explored by outstanding scientists (see [4, 16, 18, 19, 39, 40, 43, 49, 56, 57]). The authors in [4, 18, 43] carried out a research on the existence of martingale solutions to the stochastic MHD equations. The proof uses the Galerkin method with some stochastic compactness theorem

due to Prokhorov and Skorokhod. Sundar [49] has demonstrated the existence and uniqueness of solutions of the stochastic 2D MHD system orchestrated by a multiplicative or additive fractional Brownian noise. The proof uses the method of local monotonicity which was defined by Menaldi and Sritharan [39]. Motyl [40] demonstrated the existence of martingale solutions of a stochastic hydrodynamic-type evolution equations conducted by Lévy noise in 3D unbounded domains. The overall framework is applied to the stochastic Navier-Stokes, magnetohydrodynamic, and the Boussinesq equations. The authors in [56] laid the existence and uniqueness of strong solutions for the stochastic three dimensional MHD equations with multiplicative noise.

The analysis of the interaction of electromagnetic field with the mixture of two incompressible, immiscible fluids is of importance value in engineering, like MHD generators and pump accelerators (see [21, 27, 50]). In fact in MHD generators and pump accelerators, some experimental and analytical studies on the flow of the mixture of two immiscible fluids in a channel under an external magnetic field are carried out, [36, 37, 38, 46]. The Cahn-Hilliard-Magnetohydrodynamic system was given some introduction in [56, 59]. The system gives some description of the flow of the mixture of two incompressible, immiscible and electrically conducting fluids. The model consists of the Cahn-Hilliard equation, Navier-Stokes equations and Maxwell equations (magnetic field) which are nonlinearly coupled through convection, stresses and Lorentz forces. In [59], the authors make a proposal of a fully discrete energy stable finite element method with a semi-implicit scheme. They demonstrated that there exist subsequences of discrete solutions which converge to weak solution of the model for vanishing discretization parameters and the proof uses the stability of the scheme and the compactness method. In [56], the authors prove the existence, uniqueness and regularity of strong solutions to a stochastic 2D Cahn-Hilliard-Magnetohydrodynamic model, using the Galerkin method. The Cahn-Hilliard-Navier-Stokes system is a smooth interface model for incompressible isothermal mixture of two immiscible fluids and consists of the Navier-Stokes system coupled with a convective Cahn-Hilliard equations for the relative concentration difference of both fluids (see [3, 14, 22, 23, 24, 31, 51]).

The regularity of the solution for a hydrodynamic system is important in proving the ergodicity of the equations and the error estimates of the discrete time scheme of these equations. The study of the regularity and the numerical approximations of the Cahn-Hilliard, Cahn-Hilliard-Navier-Stokes, Allen-Cahn-Navier-Stokes, and Cahn-Hilliard-MHD equations have been analyzed in ([22, 24, 51, 52, 53, 54, 56]). Moreover, [53] shows a new regularity result for the weak solutions for the Allen-Cahn-Navier-Stokes equations. In [56], the authors study the regularity of the solution of the 2D stochastic Cahn-Hilliard-MHD model. The proof uses the Itô Lemma and some a priori estimates.

Stochastic partial differential equations aim at serving a mathematical model for systems with two types of forces, one deterministic and the other random. These equations permit to take into account a random noise in the evolution of a phenomenon. This may reflect for example some environmental effects on the phenomena, some external random forces, etc. With regards to the literature consulted, this is the ever first study that introduces the analysis of the stochastic version of the Allen-Cahn-Magnetohydrodynamic (AC-MHD) system.

This paper aims at analysing a class of stochastic 2D Allen-Cahn-Magnetohydrodynamic model. We demonstrate the existence, uniqueness and regularity of strong solutions and use a Galerkin approximation, the stopping time technique and some weak convergence principles in functional analysis for the proof of the existence and regularity of strong solution. Let's note that the stochastic Allen-Cahn-Navier-Stokes system was scrutinised in

[52, 53, 54].

The article is organised as follows. In the upcoming section, we elaborate on the 2D stochastic Allen-Cahn-MHD model and the functional framework. In Section 3, we proceed to an introduction of a Galerkin scheme and some a priori estimates of its solution. Furthermore, we demonstrate the existence and uniqueness of the strong solution for the stochastic Allen-Cahn-Magnetohydrodynamic system. In addition, we demonstrate that the Galerkin solution leads to the unique strong solution. In Section 4, we prove a regularity result for the solution.

2 A stochastic AC-MHD model and its mathematical setting

2.1 Governing equations

We assume that a finite time horizon $T > 0$ is given and the domain $\mathcal{M}$ of the fluid mixture is a bounded domain of $\mathbb{R}^2$ with smooth boundary.

Then, the following system is considered:

$$\left\{ \begin{array}{l} \frac{\partial v}{\partial t} - \nu_1 \Delta v + (v \cdot \nabla)v - \mathcal{K} \mu \nabla \Psi + \nabla p - (d \cdot \nabla)d = f_1(t, v) + f_2(t, v) \dot{W}_t, \\ \frac{\partial d}{\partial t} + \nu \operatorname{curl}(\operatorname{curl} d) + (v \cdot \nabla)d = (d \cdot \nabla)v, \\ \frac{\partial \Psi}{\partial t} + \nu_3 \mu + v \cdot \nabla \Psi = 0, \\ \mu = -\nu_2 \Delta \Psi + \alpha f(\Psi), \\ \operatorname{div}(v) = \operatorname{div}(d) = 0, \end{array} \right. \quad (1)$$

in $\mathcal{M} \times [0, T]$.

In (1), the unknown functions are the random velocity of the fluid mixture, its pressure p , the magnetic field d and the order (phase) parameter Ψ . The external force $f_1(t, v)$ is provided and the term $f_2(t, v) \dot{W}_t$ denotes random external forces that eventually relies on v , where $\dot{W}_t$ stands for the derivative in accordance to the time of a cylindrical Wiener process on a separable Hilbert space $\mathbb{K}$. The quantity μ is the chemical potential of the binary mixture which is provided by the variational derivative of the following free energy functional.

$$\mathcal{F}(\Psi) = \int_{\mathcal{M}} \left(\frac{\nu_2}{2} |\nabla \Psi|^2 + \alpha F(\Psi) \right) ds, \quad \text{where } e.g., \quad F(x) = \int_0^x f(\zeta) d\zeta. \quad (2)$$

The constants $\nu_1 > 0$, $\nu > 0$, $\nu_3 > 0$ and $\mathcal{K} > 0$ represent respectively the kinematic viscosity of the fluid, the magnetic resistivity, the mobility constant and the coefficient of capillarity. However ν_2 and $\alpha > 0$ are two physical quantities which characterize the interaction between the two phases. More precisely, ν_2 has a link with the area of separation of both fluids. The potential F of logarithmic type is an important example and is in certain time substituted with an approximate polynomial of the form

$$F(r) = \gamma_1 r^4 - \gamma_2 r^2, \quad \text{where } \gamma_1, \gamma_2 \text{ are positive values.} \quad (3)$$

It is worth to be noted as in [24], that the first equation of the model can take the form:

$$\frac{\partial v}{\partial t} - \nu_1 \Delta v + (v \cdot \nabla)v - (d \cdot \nabla)d + \nabla \tilde{p} = -\mathcal{K} \operatorname{div}(\nabla \Psi \otimes \nabla \Psi) + f_1(t, v) + f_2(t, v) \dot{W}_t,$$

where $\tilde{p} = p - \mathcal{K} \left(\frac{\nu_2}{2} |\nabla \Psi|^2 + \alpha F(\Psi) \right)$, since $\mathcal{K} \mu \nabla \Psi = \nabla \left(\mathcal{K} \left(\frac{\nu_2}{2} |\nabla \Psi|^2 + \alpha F(\Psi) \right) \right) - \mathcal{K} \operatorname{div}(\nabla \Psi \otimes \nabla \Psi)$.

The tensor product $\nabla \Psi \otimes \nabla \Psi$ is the contribution which highlights the capillary forces coming from the tension on the boundary zone of the fluids.

For a vector-valued function $u = (u_1, u_2)$ on $\mathcal{M}$, define $\operatorname{curl}(u) = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}$, and for a scalar-valued function

Ψ on $\mathcal{M}$, $\operatorname{curl}(\Psi) = \left(\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x} \right)$.

Also we know that $\operatorname{curl}(\operatorname{curl} u) = \operatorname{grad}(\operatorname{div} u) - \Delta u$ is valid in the two dimensions.

System (1) is combined with the following initial values and boundary conditions:

$$\begin{cases} \partial_\eta \Psi = \partial_\eta \Delta \Psi = 0 & \text{on } \partial \mathcal{M} \times (0, T), \\ v = 0 & \text{on } \partial \mathcal{M}, \\ \operatorname{curl}(d) = 0 & \text{on } \partial \mathcal{M}, \\ d \cdot \eta = 0 & \text{on } \partial \mathcal{M}, \\ u(0, x) = u_0(x); \quad d(0, x) = d_0(x); \quad \Psi(0, x) = \Psi_0(x) & \text{in } \mathcal{M}, \end{cases} \quad (4)$$

where $\partial \mathcal{M}$ is the boundary of $\mathcal{M}$ and η is the outward normal to $\partial \mathcal{M}$.

With regard to the first equation in (4), we imply the conservation of the following quantity

$$\langle \Psi(t) \rangle = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} \Psi(x, t) dx, \quad (5)$$

where $|\mathcal{M}|$ is the Lebesgue measure of $\mathcal{M}$. More exactly, we have

$$\langle \Psi(t) \rangle = \langle \Psi(0) \rangle = M_0, \quad \forall t \geq 0. \quad (6)$$

We assume from it that it can sometimes be supposed that the average of Ψ is null at the initial moment, starting from a displacement of the field of parameter of order. Hence, it will be zero for all positive instants.

So, we suppose that

$$\langle \Psi(t) \rangle = \langle \Psi(0) \rangle = 0, \quad \forall t \geq 0. \quad (7)$$

Remark 2.1. *The conservation of mass given by (7) does not hold if a stochastic perturbation of the Cahn-Hilliard system is considered.*

2.2 Mathematical setting

We note from [25] a weak formulation of (1). We also assume that the domain $\mathcal{M}$ of the fluid mixture is a bounded domain of $\mathbb{R}^2$, with smooth boundary (for example, of class C^3) and that $f \in C^2(\mathbb{R})$ satisfies

$$\begin{cases} \lim_{|r| \rightarrow +\infty} f'(r) > 0, \\ |f''(r)| \leq c_f(1 + |r|^2), \quad \forall r \in \mathbb{R}, \end{cases} \quad (8)$$

where c_f is a positive quantity.

Let us begin with the selection of the function spaces of the model (1).

If X is a real Hilbert space endowed with the scalar product $(\cdot, \cdot)_X$, we consider as notation of its associated norm $|\cdot|_X$, as well as X^* will designate its dual.

We set

$$\mathcal{V}_1 = \{u \in (C_c^\infty(\mathcal{M}))^2 : \operatorname{div}(u) = 0 \text{ in } \mathcal{M}\}.$$

Let's design by H_1 the closure de $\mathcal{V}_1$ in $(L^2(\mathcal{M}))^2$, and V_1 the closure de $\mathcal{V}_1$ in $(H_0^1(\mathcal{M}))^2$.

We denote the scalar product in H_1 by $(\cdot, \cdot)_{L^2}$ and its associated norm by $|\cdot|_{L^2}$, respectively.

Moreover, the space V_1 is accompanied by the scalar product

$$((u, v)) = \sum_{i=1}^2 (\partial_{x_i} u, \partial_{x_i} v)_{L^2},$$

and by the norm

$$||u|| = ((u, u))^{1/2}.$$

We now consider the operator A_0 defined by

$$A_0 u = -\mathcal{P} \Delta u, \quad \forall u \in D(A_0) = H^2(\mathcal{M}) \cap V_1,$$

where $\mathcal{P}$ is the Leray-Helmholtz projector in $L^2(\mathcal{M})$ onto H_1 .

We also set

$$\mathcal{V}_2 = \{d \in (C^\infty(\bar{\mathcal{M}}))^2 : \operatorname{div}(d) = 0 \text{ and } d \cdot \eta = 0 \text{ in } \partial \mathcal{M}\},$$

and denote by H_2 and V_2 the closure of $\mathcal{V}_2$ in $(L^2(\mathcal{M}))^2$ and $(H^1(\mathcal{M}))^2$, respectively.

H_2 is endowed with the scalar product $(\cdot, \cdot)_{L^2}$ and the corresponding norm $|\cdot|_{L^2}$. We have $H_1 = H_2$ (see [56]).

V_2 is endowed with the scalar product $((\Psi, \psi))_{V_2} = (\operatorname{curl} \Psi, \operatorname{curl} \psi)_{L^2}$ and its corresponding norm $||\Psi||_{V_2} = \sqrt{((\Psi, \Psi))_{V_2}}$, which is equivalent to the norm in $H^1(\mathcal{M})$.

We introduce the operators $A_1 \in \mathcal{L}(V_2, V_2^*)$ defined by $\langle A_1 u, v \rangle = ((u, v))_2$ for all $u, v \in V_2$.

The operator A_1 can also be defined as an unbounded operator generated by the boundary value problem

$$\begin{cases} \operatorname{curl}(\operatorname{curl} d) = g & \text{in } \mathcal{M}, \\ \operatorname{div}(d) = 0 & \text{in } \mathcal{M}, \\ d \cdot \eta = 0, \quad \operatorname{curl}(d) = 0 & \text{on } \partial \mathcal{M}, \end{cases} \quad (9)$$

with domain

$$D(A_1) = \{C \in V_2 : A_1 C \in H_2\}.$$

For more details about the operator A_1 , we refer the reader to [17, 36].

It is logical that from the relation (8), we can choose $\gamma > 0$ such as $\lim_{|r| \rightarrow +\infty} f'(r) > 2\gamma > 0$.

By analogy, we introduce the linear positive unbounded operator A_γ on $L^2(\mathcal{M})$ by

$$A_\gamma \Psi = -\Delta \Phi + \gamma \Psi, \quad \forall \Psi \in D(A_\gamma) = \{\rho \in H^2(\mathcal{M}), \quad \partial_\eta \rho = 0 \text{ on } \partial \mathcal{M}\}. \quad (10)$$

Note that A_γ^{-1} is a compact linear operator on $L^2(\mathcal{M})$ and $|A_\gamma \cdot|_{L^2}$ is a norm on $D(A_\gamma)$ that is equivalent to the H^2 -norm.

As in [53], we introduce the bilinear operators B, B_1 (with their associated trilinear forms $\mathbf{b}_0, \mathbf{b}_1$), and R_0 , which are defined from $D(A_0) \times D(A_0)$ into H_1 , $D(A_0) \times D(A_\gamma)$ into $L^2(\mathcal{M})$, and $(L^2(\mathcal{M}))^2 \times D(A_\gamma^{3/2})$ into H_1 , respectively. More exactly, we set

$$\langle B(u, v), w \rangle = \int_{\mathcal{M}} [(u \cdot \nabla) v] \cdot w dx = \mathbf{b}_0(u, v, w), \quad \forall u, v, w \in D(A_0),$$

$$\langle B(d, d), w \rangle = \int_{\mathcal{M}} [(d \cdot \nabla) d] \cdot w dx, \quad \forall w \in D(A_0), \quad d \in D(A_1),$$

$$\begin{aligned}
\langle B(d, v), \beta \rangle &= \int_{\mathcal{M}} [(d \cdot \nabla) v] \cdot \beta dx, \quad \forall v \in D(A_0), \quad d, \beta \in D(A_1), \\
\langle B(v, d), \beta \rangle &= \int_{\mathcal{M}} [(v \cdot \nabla) d] \cdot \beta dx, \quad \forall v \in D(A_0), \quad d, \beta \in D(A_1), \\
\langle B_1(u, \Psi), \rho \rangle &= \int_{\mathcal{M}} [(u \cdot \nabla) \Psi] \rho dx = b_1(u, \Psi, \rho), \quad \forall u \in D(A_0), \quad \Psi, \rho \in D(A_\gamma), \\
\langle R_0(\mu, \Psi), w \rangle &= \int_{\mathcal{M}} \mu [\nabla \Psi \cdot w] dx = b_1(w, \Psi, \mu), \quad \forall w \in D(A_0), \quad \Psi \in D(A_\gamma^{3/2}), \quad \mu \in L^2(\mathcal{M}).
\end{aligned}$$

Note that

$$\begin{aligned}
R_0(\mu, \Psi) &= \mathcal{P}(\mu \nabla \Psi). \\
\langle B(u, v), w \rangle &= -\langle B(u, w), v \rangle, \quad \forall u, v, w \in V_1, \\
\langle B(u, v), v \rangle &= 0, \quad \forall u, v \in V_1,
\end{aligned}$$

where $\mathcal{P}$ is the Leray-Helmholtz projector in $L^2(\mathcal{M})$.

We set

$$\mathbb{Y} = H_1 \times H_2 \times H^1(\mathcal{M}). \quad (11)$$

The space $\mathbb{Y}$ is a complete metric space with respect to the norm

$$|(v, d, \Psi)|_{\mathbb{Y}}^2 = |v|_{L^2}^2 + |d|_{L^2}^2 + \nu_2(|\nabla \Psi|_{L^2}^2 + \gamma|\Psi|_{L^2}^2). \quad (12)$$

We define the Hilbert space $\mathbb{V}$ by

$$\mathbb{V} = V_1 \times V_2 \times D(A_\gamma), \quad (13)$$

endowed with the scalar products whose associated norm is

$$|(v, d, \Psi)|_{\mathbb{V}}^2 = \|v\|^2 + |\operatorname{curl}(d)|_{L^2}^2 + |A_\gamma \Psi|_{L^2}^2. \quad (14)$$

We also set

$$f_\gamma(r) = f(r) - \alpha^{-1} \nu_2 \gamma r,$$

and observe that f_γ still satisfies (8) with γ in place of 2γ since $\nu_2 \leq \alpha$. Also its primitive $F_\gamma(r) = \int_0^r f_\gamma(\zeta) d\zeta$ is bounded from below.

As in [25, 53], we derive that B , B_1 and R_0 verify the below mentioned estimates

$$\begin{aligned}
|B(u, v)|_{V_1^*} &\leq c|u|_{L^2}^{1/2} \|u\|^{1/2} \|v\|, \quad \forall u, v \in V_1, \\
|B(u, d)|_{V_2^*} &\leq c|u|_{L^2}^{1/2} \|u\|^{1/2} \|d\|, \quad \forall u \in V_1, \quad d \in D(A_1) \\
|B(d, u)|_{V_2^*} &\leq c|d|_{L^2}^{1/2} \|d\|^{1/2} \|u\|, \quad \forall u \in V_1, \quad d \in D(A_1) \\
|B_1(v, \Psi)|_{L^2} &\leq c|v|_{L^2}^{1/2} \|v\|^{1/2} \|\Psi\|^{1/2} |A_\gamma \Psi|_{L^2}^{1/2}, \quad \forall v \in V_1, \quad \Psi \in D(A_\gamma), \\
|R_0(A_\gamma \Psi, \rho)|_{V_1^*} &\leq c\|\rho\|^{1/2} |A_\gamma \rho|_{L^2}^{1/2} |A_\gamma \Psi|_{L^2}, \quad \forall \Psi, \rho \in D(A_\gamma).
\end{aligned} \quad (15)$$

We consider a positive value c which depends on the domain $\mathcal{M}$. In order to simplify the notations, we set (without loss of generality) $\nu_1 = \nu_2 = \nu_3 = \nu = \alpha = \mathcal{K} = 1$.

Using the above mentioned notations, (1) can be written as follows

$$\left\{ \begin{array}{l} \frac{dv}{dt} + A_0 v + B(v, v) - R_0(A_\gamma \Psi, \Psi) - B(d, d) = f_1(t, v) + f_2(t, v) \dot{W}_t \quad \text{in } V_1^*, \\ \frac{d(d)}{dt} + [B(v, d) - B(d, v) + A_1 d] = 0 \quad \text{in } V_2^*, \\ \frac{d\Psi}{dt} + \mu + B_1(v, \Psi) = 0 \quad \text{in } V_3^*, \\ \mu = A_\gamma \Psi + f_\gamma(\Psi), \\ (v, d, \Psi)(0) = (v_0, d_0, \Psi_0). \end{array} \right. \quad (16)$$

From Definition 1 of [56], it turns out that system (16) is equivalent to system (17) below, which is its integral form

$$\left\{ \begin{array}{l} v(t) + \int_0^t (A_0 v(s) + B(v(s), v(s)) - B(d(s), d(s))) ds = v_0 + \int_0^t R_0(A_\gamma \Psi(s), \Psi(s)) ds \\ \quad + \int_0^t f_1(s, v(s)) ds + \int_0^t f_2(s, v(s)) dW_s, \\ d(t) + \int_0^t [B(v(s), d(s)) - B(d(s), v(s)) + A_1 d(s)] ds = d_0, \\ \Psi(t) + \int_0^t (\mu(s) + B_1(v(s), \Psi(s))) ds = \Psi_0, \\ \mu = A_\gamma \Psi + f_\gamma(\Psi), \end{array} \right. \quad (17)$$

$\mathbb{P}$ -a.s., and for all $t \in [0, T]$.

Remark 2.2. In the weak formulation (16), $A_\gamma \Psi \nabla \Psi$ replaces the term $\mu \nabla \Psi$ (see [53]).

Firstly, we recall from [52] some notations and very important results related to the nonlinear coupling operator R_0 (see propositions 2.1, 2.2 and 2.3 below). To satisfy the expectancies of readers, the proofs of these results will be repeated here.

Now we introduce the trilinear form r_1 given by

$$r_1(\Psi, \psi, v) = - \sum_{i,j}^2 \int_{\mathcal{M}} \frac{\partial \Psi}{\partial x_i} \frac{\partial \psi}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx, \quad \forall \Psi, \psi \in W^{1,4}, \quad v \in V_1. \quad (18)$$

Proposition 2.1. There exists a constant $c > 0$ such as

$$|r_1(\Psi, \psi, v)| \leq c \|\Psi\|^{1/2} |A_\gamma \Psi|_{L^2}^{1/2} \|\psi\|^{1/2} |A_\gamma \psi|_{L^2}^{1/2} \|v\|, \quad \forall \Psi, \psi \in D(A_\gamma), \quad v \in V_1. \quad (19)$$

Proof. From (18), we easily derive that for all $\Psi, \psi \in D(A_\gamma), v \in V_1$, we have

$$|r_1(\Psi, \psi, v)| \leq \left| \sum_{i,j=1}^2 \int_{\mathcal{M}} D_i \Psi (D_j \psi) D_j v_i dx \right| \leq c |\nabla \Psi|_{L^4} |\nabla \psi|_{L^4} |\nabla v|_{L^2} \leq c \|\Psi\|^{1/2} |A_\gamma \Psi|_{L^2}^{1/2} \|\psi\|^{1/2} |A_\gamma \psi|_{L^2}^{1/2} \|v\|, \quad (20)$$

and (19) is proved. $\square$

Proposition 2.2. There exists a bilinear operator R_1 defined on $D(A_\gamma) \times D(A_\gamma)$ with values in V_1^* such as

$$\langle R_1(\Psi, \psi), v \rangle = r_1(\Psi, \psi, v), \quad \forall \Psi, \psi \in D(A_\gamma), \quad v \in V_1. \quad (21)$$

Furthermore, there exists a constant $c > 0$ such as

$$\|R_1(\Psi, \psi)\|_{V_1^*} \leq c \|\Psi\|^{1/2} |A_\gamma \Psi|_{L^2}^{1/2} \|\psi\|^{1/2} |A_\gamma \psi|_{L^2}^{1/2}, \quad \forall \Psi, \psi \in D(A_\gamma). \quad (22)$$

Proof. The first part of the proposition is linked to the fact that for any $v \in V_1$, the mapping $r_1(\cdot, \cdot, v)$ defined on $D(A_\gamma) \times D(A_\gamma)$ with values in $\mathbb{R}$ is continuous. The estimates (22) also follow directly from (19). $\square$

Proposition 2.3. *For all $v \in V_1$ and $\Psi \in D(A_\gamma)$, we have*

$$\langle B_1(v, \Psi), A_\gamma \Psi \rangle = -\langle R_1(\Psi, \Psi), v \rangle. \quad (23)$$

Proof. In order to simplify the notations, we assume the summation over repeated indexes. Considering the fact that $\operatorname{div}(v) = 0$ as well as the boundary conditions, we deduce by integration by part that

$$\begin{aligned} \langle B_1(v, \Psi), -A_\gamma \Psi \rangle &= \int_{\mathcal{M}} v_i \frac{\partial \Psi}{\partial x_i} \frac{\partial^2 \Psi}{\partial x_j \partial x_j} dx = - \int_{\mathcal{M}} \frac{\partial v_i}{\partial x_j} \frac{\partial \Psi}{\partial x_i} \frac{\partial \Psi}{\partial x_j} dx - \int_{\mathcal{M}} v_i \frac{\partial^2 \Psi}{\partial x_i \partial x_j} \frac{\partial \Psi}{\partial x_j} dx \\ &= - \int_{\mathcal{M}} \frac{\partial v_i}{\partial x_j} \frac{\partial \Psi}{\partial x_i} \frac{\partial \Psi}{\partial x_j} dx - \frac{1}{2} \int_{\mathcal{M}} v_i \frac{\partial |\nabla \Psi|^2}{\partial x_i} dx = - \int_{\mathcal{M}} \frac{\partial v_i}{\partial x_j} \frac{\partial \Psi}{\partial x_i} \frac{\partial \Psi}{\partial x_j} dx = r_1(\Psi, \Psi, v) = \langle R_1(\Psi, \Psi), v \rangle. \end{aligned} \quad (24)$$

Note that in (24), we use the fact that $\langle B_1(v, \Psi), f_\gamma(\Psi) \rangle = 0$. $\square$

Remark 2.3. *We recall from [54] that*

$$\operatorname{div}(\nabla \Psi \otimes \nabla \Psi) = \nabla \left(\frac{1}{2} |\nabla \Psi|^2 + \alpha F_\gamma(\Psi) \right) - \mu \nabla \Psi. \quad (25)$$

It is clear that $\operatorname{div}(\nabla \Psi \otimes \nabla \psi) \in L^2(\mathcal{M})$, $\forall \Psi \in D(A_\gamma), \psi \in D(A_\gamma^{3/2})$.

Furthermore

$$R_1(\Psi, \psi) = \mathcal{P}_1[\operatorname{div}(\nabla \Psi \otimes \nabla \psi)], \quad \forall \Psi \in D(A_\gamma), \quad \psi \in D(A_\gamma^{3/2}). \quad (26)$$

It comes from (25)-(26) that

$$\begin{aligned} R_1(\Psi, \Psi) &= \mathcal{P}_1[\operatorname{div}(\nabla \Psi \otimes \nabla \Psi)] = \mathcal{P}_1 \left[\nabla \left(\frac{1}{2} |\nabla \Psi|^2 + \alpha F_\gamma(\Psi) \right) \right] - \mathcal{P}_1[\mu \nabla \Psi] \\ &= -\mathcal{P}_1[\mu \nabla \Psi] = R_0(\mu, \Psi) = R_0(A_\gamma \Psi, \Psi). \end{aligned} \quad (27)$$

Remark 2.4. *It comes from (19) and (27) that*

$$\|R_0(A_\gamma \Psi, \Psi)\|_{V_1^*} = \|R_1(\Psi, \Psi)\|_{V_1^*} \leq c \|\Psi\| \|A_\gamma \Psi\|_{L^2}, \quad \forall \Psi \in D(A_\gamma). \quad (28)$$

Assumption 1. Suppose that f_1 and f_2 are Lipschitz and sublinear functions from $\Omega \times (0, T) \times H_1$ into V_1^* and from $\Omega \times (0, T) \times H_1$ to $\mathcal{L}^2(\mathbb{K}; H_1)$, respectively. That is, for all $v_1, v_2 \in V_1$, $f_1(\cdot, v_1)$ and $f_2(\cdot, v_1)$ are $\mathcal{F}_t$ -progressively measurable, and $d\mathbb{P} \times dt$ -a.e. in $\Omega \times (0, T)$, we have

$$\begin{aligned} \|f_1(t, v_1) - f_1(t, v_2)\|_{V_1^*} &\leq L_1 |v_1 - v_2|_{L^2}, \\ f_1(t, 0) &\in M_{\mathcal{F}_t}^2(0, T; V_1^*), \\ \|f_2(t, v_1) - f_2(t, v_2)\|_{\mathcal{L}^2(\mathbb{K}; H_1)} &\leq L_2 |v_1 - v_2|_{L^2}, \\ f_2(t, 0) &\in M_{\mathcal{F}_t}^2(0, T; \mathcal{L}^2(\mathbb{K}; H_1)). \end{aligned} \quad (29)$$

Subsequently, we do consider that

$$(v_0, d_0, \Psi_0) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{Y}). \quad (30)$$

Thus for all $(w, \beta, \Psi) \in \mathbb{Y}$, we set

$$\mathcal{E}(w, \beta, \Psi) = |w|_{L^2}^2 + \|\beta\|^2 + \|\Psi\|^2 + 2\langle F_\gamma(\Psi), 1 \rangle + c_1, \quad (31)$$

where $c_1 > 0$ is a constant large enough and independent on (w, β, Ψ) such that $\mathcal{E}(w, \beta, \Psi)$ is no-negative (note that F is bounded from below). We can verify that there exists a monotone non-decreasing function (independent on time and the initial condition) such as

$$|(w, \beta, \Psi)|_{\mathbb{Y}}^2 \leq \mathcal{E}(w, \beta, \Psi) \leq \mathcal{Q}_0(|(w, \beta, \Psi)|_{\mathbb{Y}}^2), \quad (w, \beta, \Psi) \in \mathbb{Y}. \quad (32)$$

Definition 2.1. A variational solution to (16) or (17) is a process $(v, d, \Psi) \in M_{\mathcal{F}_t}^2(0, T; \mathbb{Y}) \cap L^2(\Omega, L^\infty(0, T; \mathbb{Y}))$, weakly continuous with values in $\mathbb{Y}$ such as (17) is verified in V^* , $\mathbb{P}$ -a.s., for all $t \in [0, T]$.

3 Existence and uniqueness of solutions of the model

In this section, we demonstrate the existence and uniqueness of variational solution to (16).

Proposition 3.1. If (v, d, Ψ) is a variational solution to (16) or (17), then $(v, d, \Psi) \in L^2(\Omega, C[0, T; \mathbb{Y}))$, and

$$\begin{aligned} \mathcal{E}(v, d, \Psi)(t) + 2 \int_0^t (||v(s)||^2 + ||d(s)||^2 + |\mu(s)|_{L^2}^2) ds &= \mathcal{E}(v_0, d_0, \Psi_0) + 2 \int_0^t \langle f_1(s, v(s)), v(s) \rangle ds \\ + 2 \int_0^t \langle f_2(s, v(s)), v(s) dW_s \rangle + \int_0^t ||f_2(s, v(s))||_{L^2(\mathbb{K}, H_1)}^2 ds, \quad \forall t \in [0, T], \\ \mathbb{E} \int_0^t \langle f_2(s, v(s)), v(s) dW_s \rangle &= 0, \quad \forall t \in [0, T]. \end{aligned} \quad (33)$$

Proof. From (16)₁ and the Itô's formula, we deduce that

$$\begin{aligned} |v(t)|_{L^2}^2 + 2 \int_0^t \langle A_0 v, v \rangle ds &= |v_0|_{L^2}^2 + 2 \int_0^t \langle R_0(A_\gamma \Psi, \Psi), v \rangle ds + 2 \int_0^t \langle B(d, d), v \rangle ds \\ + 2 \int_0^t \langle f_1(s, v(s)), v(s) \rangle ds + 2 \int_0^t \langle f_2(s, v(s)), v(s) dW_s \rangle + \int_0^t ||f_2(s, v(s))||_{L^2(\mathbb{K}, H_1)}^2 ds, \quad \forall t \in [0, T]. \end{aligned} \quad (34)$$

Now, we take the scalar product in H_2 of (16)₂ with b to derive that

$$||d||^2 + 2 \int_0^t \langle A_1 d, d \rangle ds + 2 \int_0^t \langle B(d, d), v \rangle ds = |d_0|_{L^2}^2. \quad (35)$$

We multiply (16)₃ by μ and using (16)₄ to derive that

$$\begin{aligned} \left\langle \frac{d\Psi}{dt}, \mu \right\rangle_{L^2} + \langle \mu, \mu \rangle_{L^2} + \langle B_1(v, \Psi), \mu \rangle &= 0, \\ \left\langle \frac{d\Psi}{dt}, \mu \right\rangle_{L^2} &= \left\langle \frac{d\Psi}{dt}, A_\gamma \Psi \right\rangle + \left\langle \frac{d\Psi}{dt}, f_\gamma(\Psi) \right\rangle = \frac{1}{2} \frac{d}{dt} ||\Psi||^2 + \left\langle \frac{dF_\gamma(\Psi)}{dt}, 1 \right\rangle. \end{aligned}$$

Then

$$d(||\Psi||^2 + 2\langle F_\gamma(\Psi), 1 \rangle) + 2(|\mu|_{L^2}^2 + b_1(v, \Psi, \mu))dt = 0. \quad (36)$$

Integrating (36) from 0 to t , we obtain

$$||\Psi||^2 + 2\langle F_\gamma(\Psi), 1 \rangle + 2 \int_0^t (|\mu|_{L^2}^2 + b_1(v, \Psi, \mu))dt = |\Psi_0|_{L^2}^2. \quad (37)$$

Adding the resulting equation to (34), (35), and (37) gives

$$\begin{aligned} |v(t)|_{L^2}^2 + 2 \int_0^t ||v(s)||^2 ds + ||d||^2 + ||\Psi||^2 + 2\langle F_\gamma(\Psi), 1 \rangle + 2 \int_0^t (|\mu|_{L^2}^2 + b_1(v, \Psi, \mu))dt &+ 2 \int_0^t \langle A_1 d, d \rangle ds \\ + 2 \int_0^t \langle B(d, d), v \rangle ds &= |v_0|_{L^2}^2 + |d_0|_{L^2}^2 + |\Psi_0|_{L^2}^2 + 2 \int_0^t \langle R_0(A_\gamma \Psi, \Psi), v \rangle ds + 2 \int_0^t \langle B(d, d), v \rangle ds \\ + 2 \int_0^t \langle f_1(s, v(s)), v(s) \rangle ds + 2 \int_0^t \langle f_2(s, v(s)), v(s) dW_s \rangle + \int_0^t ||f_2(s, v(s))||_{L^2(\mathbb{K}, H_1)}^2 ds, \quad \forall t \in [0, T]. \end{aligned} \quad (38)$$

And using the fact that (see [25])

$$\langle R_0(A_\gamma \Psi, \Psi), v \rangle = \langle B_1(v, \Psi), \mu \rangle = \langle B_1(v, \Psi), A_\gamma \Psi \rangle,$$

we have

$$\begin{aligned} \mathcal{E}(v, d, \Psi) + 2 \int_0^t \|v(s)\|^2 + \|d(s)\|^2 + |\mu(s)|_{L^2}^2 ds &= \mathcal{E}(v_0, d_0, \Psi_0) + 2 \int_0^t \langle f_1(s, v(s)), v(s) \rangle ds \\ &+ 2 \int_0^t \langle f_2(s, v(s)), v(s) dW_s \rangle + \int_0^t \|f_2(s, v(s))\|_{\mathcal{L}^2(\mathbb{K}; H_1)}^2 ds, \quad \forall t \in [0, T]. \end{aligned} \quad (39)$$

If (v, d, Ψ) is a solution to (16) or (17), then $B(v, v), R_0(A_\gamma \Psi, \Psi), B(d, d) \in M_{\mathcal{F}_t}^2(0, T; V_1^*)$, $\mu + B_1(v, \Psi) \in M_{\mathcal{F}_t}^2(0, T; H_3)$, $A_0 v + f_1(t, v) \in M_{\mathcal{F}_t}^2(0, T; V_1^*)$, $f_2(t, v) \in M_{\mathcal{F}_t}^2(0, T; \mathcal{L}^2(\mathbb{K}, H_1))$, and $B(v, d) - B(d, v) + A_1 b \in M_{\mathcal{F}_t}^2(0, T; V_2^*)$. Thus, with regards with the theorem 3.2 of [41], (v, d, Ψ) is $\mathbb{P}$ -a.s. continuous with values in $\mathbb{Y}$. Therefore (33) holds.

Furthermore, we note that (16)₂ follows due to $f_2(t, v) \in M_{\mathcal{F}_t}^2(0, T; \mathcal{L}^2(\mathbb{K}, H_1))$ and $v \in L^2(\Omega, L^\infty(0, T; H_1))$.

Proposition 3.2. *There exists at most one variational solution to (16).*

Proof. Let $(v_1, d_1, \Psi_1), (v_2, d_2, \Psi_2)$ be variational solutions to (16). Let $(w, \beta, \psi, \mu) = (v_1, d_1, \Psi_1, \mu_1) - (v_2, d_2, \Psi_2, \mu_2)$, $\bar{\mu} = \mu - \langle \mu \rangle$ and $\mathbb{E} \int_0^t \langle f_2(s, v_1) - f_2(s, v_2), w dW_s \rangle = 0$.

We have

$$\begin{cases} \frac{dv_1}{dt} + A_0 v_1 + B(v_1, v_1) - R_0(A_\gamma \Psi_1, \Psi_1) - B(d_1, d_1) = f_1(t, v_1) + f_2(t, v_1) \dot{W}_t, \\ \frac{d(d_1)}{dt} + [B(v_1, d_1) - B(d_1, v_1) + A_1 d_1] = 0, \\ \frac{d\Psi_1}{dt} + \mu_1 + B_1(v_1, \Psi_1) = 0, \\ \mu_1 = A_\gamma \Psi_1 + f_\gamma(\Psi_1), \\ (v_1, b_1, \Psi_1)(0) = (v_1(0), b_1(0), \Psi_1(0)), \end{cases} \quad (40)$$

and

$$\begin{cases} \frac{dv_2}{dt} + A_0 v_2 + B(v_2, v_2) - R_0(A_\gamma \Psi_2, \Psi_2) - B(d_2, d_2) = f_1(t, v_2) + f_2(t, v_2) \dot{W}_t, \\ \frac{d(b_2)}{dt} + [B(v_2, d_2) - B(d_2, v_2) + A_1 d_2] = 0, \\ \frac{d\Psi_2}{dt} + \mu_2 + B_1(v_2, \Psi_2) = 0, \\ \mu_2 = A_\gamma \Psi_2 + f_\gamma(\Psi_2), \\ (v_2, b_2, \Psi_2)(0) = (v_2(0), b_2(0), \Psi_2(0)). \end{cases} \quad (41)$$

Then (w, β, ψ) is solution to the following system

$$\begin{cases} dw + [A_0 w + B(w, v_1) + B(v_2, w)]dt - [R_0(A_\gamma \psi, \Psi_1) + R_0(A_\gamma \Psi_2, \psi)]dt \\ \quad - [B(\beta, d_1) + B(d_2, \beta)]dt = [f_1(t, v_1) - f_1(t, v_2)]dt + [f_2(t, v_1) - f_2(t, v_2)]dW, \\ d\beta + [B(w, d_1) + B(v_2, \beta)]dt - [B(\beta, v_1) + B(d_2, w)]dt + A_1 \beta dt = 0, \\ d\psi + \mu dt + [B_1(w, \Psi_1) + B_1(v_2, \psi)]dt = 0, \\ \mu = A_\gamma \psi + f_\gamma(\Psi_1) - f_\gamma(\Psi_2), \\ (w, \beta, \psi)(0) = (w(0), \beta(0), \psi(0)). \end{cases} \quad (42)$$

Reasoning as in the proof of Proposition 3.1 and using Itô's formula, we deduce that

$$\begin{aligned} |w(t)|_{L^2}^2 + 2 \int_0^t (||w||^2 + \langle B(w, v_1), w \rangle) ds &= 2 \int_0^t \langle R_0(A_\gamma \Psi_2, \psi) + R_0(A_\gamma \psi, \Psi_1), w \rangle ds \\ + 2 \int_0^t \langle B(\beta, d_1), w \rangle + \langle B(d_2, \beta), w \rangle ds &+ 2 \int_0^t \langle f_1(s, v_1) - f_1(s, v_2), w \rangle ds \\ + \int_0^t ||f_2(s, v_1) - f_2(s, v_2)||_{L^2(\mathbb{K}; H_1)}^2 ds &+ 2 \int_0^t \langle f_2(s, v_1) - f_2(s, v_2), w dW_s \rangle. \end{aligned} \quad (43)$$

Taking the duality of (42)₂ with β , we have

$$||\beta||^2 + 2 \int_0^t \langle B(w, d_1), \beta \rangle ds - 2 \int_0^t \langle B(\beta, v_1) + B(d_2, w), \beta \rangle ds + 2 \int_0^t \langle A_1 \beta, \beta \rangle ds = 0. \quad (44)$$

Now we take the duality of (42)₃ and (42)₄ with $A_\gamma \psi$, and $A_\gamma \psi$ respectively, we get

$$\frac{1}{2} \frac{d}{dt} ||\psi||^2 + \langle \mu, A_\gamma \psi \rangle + b_1(v_2, \psi, A_\gamma \psi) + b_1(w, \Psi_1, A_\gamma \psi) = 0, \quad (45)$$

and

$$\langle \mu, A_\gamma \psi \rangle = |A_\gamma \psi|_{L^2}^2 + \langle f_\gamma(\Psi_1) - f_\gamma(\Psi_2), A_\gamma \psi \rangle. \quad (46)$$

Adding (45) and (46) gives

$$||\psi||^2 + 2 \int_0^t |A_\gamma \psi|_{L^2}^2 + \langle f_\gamma(\Psi_1) - f_\gamma(\Psi_2), A_\gamma \psi \rangle ds = -2 \int_0^t b_1(v_2, \psi, A_\gamma \psi) + b_1(w, \Psi_1, A_\gamma \psi) ds. \quad (47)$$

Adding (44), (47) and (43) gives

$$\begin{aligned} |w(t)|^2 + ||\beta(t)||^2 + ||\psi(t)||^2 + 2 \int_0^t (||w||^2 + |A_\gamma \psi|_{L^2}^2) ds &= 2 \int_0^t \langle (B(\beta, d_1), w) ds \\ - 2 \int_0^t \langle B(w, v_1), w \rangle ds + 2 \int_0^t \langle R_0(A_\gamma \Psi_2, \psi), w \rangle + \langle R_0(A_\gamma \psi, \Psi_1), w \rangle &- b_1(w, \Psi_1, A_\gamma \psi) - b_1(v_2, \psi, A_\gamma \psi) ds \\ - 2 \int_0^t \langle f_\gamma(\Psi_1) - f_\gamma(\Psi_2), A_\gamma \psi \rangle ds + 2 \int_0^t \langle f_1(s, v_1) - f_1(s, v_2), w \rangle ds &+ \int_0^t ||f_2(s, v_1) - f_2(s, v_2)||_{L^2(\mathbb{K}; H_1)}^2 ds \\ + 2 \int_0^t \langle f_2(s, v_1) - f_2(s, v_2), w dW_s \rangle - 2 \int_0^t \langle B(w, d_1), \beta \rangle ds &+ 2 \int_0^t \langle B(\beta, v_1), \beta \rangle ds - 2 \int_0^t ||\beta||^2 ds. \end{aligned} \quad (48)$$

We readily get

$$\begin{aligned} \langle R_0(A_\gamma \Psi_2, \psi), w \rangle + \langle R_0(A_\gamma \psi, \Psi_1), w \rangle - b_1(w, \Psi_1, A_\gamma \psi) - b_1(v_2, \psi, A_\gamma \psi) &= b_1(w, \psi, A_\gamma \Psi_2) + b_1(w, \Psi_1, A_\gamma \psi) - \\ b_1(w, \Psi_1, A_\gamma \psi) - b_1(v_2, \psi, A_\gamma \psi) &= b_1(w, \psi, A_\gamma \Psi_2) - b_1(v_2, \psi, A_\gamma \psi). \end{aligned}$$

In view of all these equalities, we finally obtain

$$\begin{aligned} |w(t)|^2 + ||\beta(t)||^2 + ||\psi(t)||^2 + 2 \int_0^t (||w||^2 + ||\beta||^2 + |A_\gamma \psi|_{L^2}^2) ds &= 2 \int_0^t \langle B(\beta, d_1), w \rangle + \langle B(\beta, v_1), \beta \rangle ds \\ - 2 \int_0^t \langle B(w, v_1), w \rangle ds + 2 \int_0^t (b_1(w, \psi, A_\gamma \Psi_2) - b_1(v_2, \psi, A_\gamma \psi)) ds &- 2 \int_0^t \langle B(w, d_1), \beta \rangle ds \\ - 2 \int_0^t \langle f_\gamma(\Psi_1) - f_\gamma(\Psi_2), A_\gamma \psi \rangle ds + 2 \int_0^t \langle f_1(s, v_1) - f_1(s, v_2), w \rangle ds &+ \int_0^t ||f_2(s, v_1) - f_2(s, v_2)||_{L^2(\mathbb{K}; H_1)}^2 ds \\ + 2 \int_0^t \langle f_2(s, v_1) - f_2(s, v_2), w dW_s \rangle. \end{aligned} \quad (49)$$

As in [25, 51, 58], it is worth recalling the following estimates:

Applying the Poincaré and Young's inequalities, it comes that

$$\begin{aligned} |\langle B(w, v_1), w \rangle| &\leq ||v_1|| ||w|_{L^4}^2 \leq c ||v_1|| ||w|_{L^2}^2 ||w|| \leq \frac{1}{2} ||w|| (2c ||v_1|| ||w|_{L^2}^2) \\ &\leq \frac{1}{2} \left(\frac{1}{2} ||w|| \right)^2 + \frac{1}{2} (2c ||v_1|| ||w|_{L^2}^2)^2 \leq \frac{1}{8} ||w||^2 + c ||v_1||^2 ||w|_{L^2}^2, \end{aligned} \quad (50)$$

$$|\langle B(\beta, v_1), \beta \rangle| + |\langle B(\beta, d_1), w \rangle| + |\langle B(w, d_1), \beta \rangle| \leq \frac{1}{8} \|\beta\|^2 + c \|v_1\|^2 \|\beta\|_{L^2}^2 + 2c \|d_1\| \|\beta\|_{L^2}^{1/2} \|\beta\|^{1/2} \|w\|_{L^2}^{1/2} \|w\|^{1/2}. \quad (51)$$

Note that

$$\begin{aligned} |\langle B(\beta, d_1), w \rangle| + |\langle B(w, d_1), \beta \rangle| &\leq 2c \|d_1\| \|w\|_{L^2}^{1/2} \|w\|^{1/2} \|\beta\|_{L^2}^{1/2} \|\beta\|^{1/2} \\ &\leq \frac{1}{2} \|\beta\|^{1/2} \left(4c \|d_1\| \|w\|_{L^2}^{1/2} \|w\|^{1/2} \|\beta\|_{L^2}^{1/2} \right) \\ &\leq \frac{1}{2} \left(\frac{1}{2} \|\beta\|^{1/2} \right)^2 + \frac{1}{2} \left(4c \|d_1\| \|w\|_{L^2}^{1/2} \|w\|^{1/2} \|\beta\|_{L^2}^{1/2} \right)^2 \\ &\leq \frac{1}{8} \|\beta\| + \frac{1}{2} \|w\| (16c \|d_1\|^2 \|w\|_{L^2} \|\beta\|_{L^2}) \\ &\leq \frac{1}{8} \|\beta\| + \frac{1}{2} \left(\frac{1}{2} \|w\| \right)^2 + \frac{1}{2} (16c \|d_1\|^2 \|w\|_{L^2} \|\beta\|_{L^2})^2 \\ &\leq \frac{1}{8} \|\beta\|^2 + \frac{1}{8} \|w\|^2 + c \|d_1\|^4 \|w\|_{L^2}^2 \|\beta\|_{L^2}^2. \end{aligned} \quad (52)$$

Similarly, we have

$$\begin{aligned} |\langle R_0(A_\gamma \Psi_2, \psi), w \rangle| &= |b_1(w, \psi, A_\gamma \Psi_2)| \leq c \|w\|_{L^2}^{1/2} \|w\|^{1/2} \|\psi\|_{H^1}^{1/2} \|A_\gamma \psi\|_{L^2}^{1/2} \|A_\gamma \Psi_2\|_{L^2} \\ &\leq \frac{1}{2} \|A_\gamma \Psi_2\|_{L^2} (2c \|w\|_{L^2}^{1/2} \|w\|^{1/2} \|\psi\|_{H^1}^{1/2} \|A_\gamma \psi\|_{L^2}^{1/2}) \\ &\leq \frac{1}{2} \left(\frac{1}{2} \|A_\gamma \Psi_2\|_{L^2} \right)^2 + \frac{1}{2} \left(2c \|w\|_{L^2}^{1/2} \|w\|^{1/2} \|\psi\|_{H^1}^{1/2} \|A_\gamma \psi\|_{L^2}^{1/2} \right)^2 \\ &\leq \frac{1}{8} \|A_\gamma \Psi_2\|_{L^2}^2 + \frac{1}{2} \|w\| (4c \|w\|_{L^2} \|\psi\|_{H^1} \|A_\gamma \psi\|_{L^2}) \\ &\leq \frac{1}{8} (\|w\|^2 + \|A_\gamma \Psi_2\|_{L^2}^2) + c \|w\|_{L^2}^2 \|\psi\|_{H^1}^2 \|A_\gamma \psi\|_{L^2}^2, \end{aligned} \quad (53)$$

$$|\langle f_\gamma(\Psi_1) - f_\gamma(\Psi_2), A_\gamma \psi \rangle| \leq \frac{1}{8} \|A_\gamma \psi\|_{L^2}^2 + Q_1(\|\Psi_1\|_{H^1}, \|\Psi_2\|_{H^1}) \|\psi\|^2, \quad (54)$$

$$|b_1(v_2, \psi, A_\gamma \psi)| \leq \frac{1}{8} \|A_\gamma \psi\|_{L^2}^2 + c \|v_2\|_{L^2}^2 \|v_2\|^2 \|\psi\|^2, \quad (55)$$

$$|\langle f_1(s, v_1) - f_1(s, v_2), w \rangle| \leq L_1 \|w\| \|w\|_{L^2} \leq \frac{1}{8} \|w\|^2 + c L_1^2 \|w\|_{L^2}^2, \quad (56)$$

$$\|f_2(s, v_1) - f_2(s, v_2)\|_{\mathcal{L}^2(\mathbb{K}; H_1)}^2 \leq L_2^2 \|w\|_{L^2}^2, \quad (57)$$

$$|\langle A_1 \beta, \beta \rangle| = \|\beta\|^2.$$

Set

$$\mathcal{Y}_2(t) = \|w(t)\|_{L^2}^2 + \|\beta(t)\|^2 + \|\psi(t)\|^2,$$

$$K_1(t) = c (\|v_1\|^2 + \|d_1\|^2 + \|\Psi_2\|^2 \|A_\gamma \Psi_2\|_{L^2}^2 + \|v_2\|_{L^2}^2 \|v_2\|^2) + Q_1(\|\Psi_1\|_{H^1}, \|\Psi_2\|_{H^1}), \quad (58)$$

$$\sigma(t) = \exp \left(-2 \int_0^t K_1(s) ds \right), \quad d\sigma(t) = -2K_1(t) \sigma(t) dt. \quad (59)$$

From (50)-(59), in view of (49), we arrive at the following inequalities

$$\begin{aligned} \sigma(t) d\mathcal{Y}_2(t) + \sigma(t) [\|w\|^2 + \|\beta\|^2 + \|A_\gamma \psi\|_{L^2}^2] dt + \mathcal{Y}_2(t) d\sigma(t) &\leq \mathcal{Y}_2(t) (2c L_1^2 + L_2^2) \sigma(t) dt \\ + 2\sigma(t) \langle f_2(t, v_1) - f_2(t, v_2), w \rangle dW_t. \end{aligned} \quad (60)$$

Hence

$$\begin{aligned} \sigma(t) \mathcal{Y}_2(t) &+ \int_0^t \sigma(s) (\|w\|^2 + \|\beta\|^2 + \|A_\gamma \psi\|_{L^2}^2) ds \\ &\leq \sigma(0) \mathcal{Y}_2(0) + (2c L_1^2 + L_2^2) \int_0^t \sigma(s) \mathcal{Y}_2(s) ds + 2 \int_0^t \sigma(s) \langle f_2(s, v_1) - f_2(s, v_2), w \rangle dW_s. \end{aligned} \quad (61)$$

To the fact that $0 < \sigma(t) \leq 1$ and the expectation of the stochastic integral in (61) vanishes, we deduce that

$$\mathbb{E}\sigma(t)\mathcal{Y}_2(t) \leq (2cL_1^2 + L_2^2)\mathbb{E} \int_0^t \sigma(s)\mathcal{Y}_2(s)ds, \quad 0 \leq t \leq T.$$

It comes from the Gronwall lemma that $\mathcal{Y}_2(t) = 0$ $\mathbb{P}$ -a.s., for all $t \in [0, T]$, which gives $(v_1, d_1, \Psi_1) = (v_2, d_2, \Psi_2)$ $\mathbb{P}$ -a.s., for all $t \in [0, T]$.

The following theorem proves the existence and uniqueness of a strong solution for problem (16) associated to the stochastic Allen-Cahn-MHD system.

Theorem 3.1. *We assume all the above mentioned and suppose that $f_1(\cdot, 0) \in L^4(\Omega, L^2(0, T, L^2(0, T; V_1^*)))$, $f_2(\cdot, 0) \in L^4(\Omega, L^2(0, T, L^2(0, T; \mathcal{L}^2(\mathbb{K}, H_1))))$, and $(v_0, d_0, \Psi_0) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}, \mathbb{Y})$ satisfies $\mathbb{E}[\mathcal{E}(v_0, d_0, \Psi_0)]^2 < \infty$. Then, there exists a unique solution $(v, d, \Psi) \in L^4(\Omega, C(0, T; \mathbb{Y})) \cap L^4(\Omega, L^2(0, T; \mathbb{V}))$.*

Moreover, the following estimate holds as follows:

$$\begin{aligned} & \mathbb{E} \left(\sup_{[0, T]} |(v, d, \Psi)|_{\mathbb{Y}}^4 \right) + \mathbb{E} \left[\left(\int_0^T \|(v, d, \Psi)\|_{\mathbb{V}}^2 dt \right)^2 \right] \leq c\mathbb{E}[\mathcal{E}(v_0, d_0, \Psi_0)]^2 \\ & + c\mathbb{E} \left[\left(\int_0^T \|f_1(t, 0)\|_{V_1^*}^2 dt \right)^2 \right] + c\mathbb{E} \left[\left(\int_0^T \|f_2(t, 0)\|_{\mathcal{L}^2(\mathbb{K}, H_1)}^2 dt \right)^2 \right], \\ & \mathbb{E} \int_0^t \langle f_2(s, v(s)), v(s) dW_s \rangle = 0. \end{aligned} \quad (62)$$

Proof. The justification follows steps a like to those in [10, 15].

Step 1: Galerkin approximation.

We define the approximate problem (63) in the finite dimensional space $\mathbb{V}_m$ of $\mathbb{V}$, having a unique solution $u_m = (v_m, d_m, \Psi_m) \in \mathbb{V}_m$, where $\mathbb{V}_m = \mathbb{Y}_m = \text{span}\{(w_1, \beta_1, \psi_1), \dots, (w_m, \beta_m, \psi_m)\}$ is the Galerkin space, which is made up of vectors $\{w_i, i = 1, 2, \dots\}$, $\{\beta_i, i = 1, 2, \dots\}$, $\{\psi_i, i = 1, 2, \dots\}$ forming an orthonormal basis of $\mathbb{V}$.

$$\begin{cases} \langle v_m(t), w_k \rangle + \int_0^t \langle A_0 v_m + B(v_m, v_m) - R_0(A_\gamma \Psi_m, \Psi_m) - B(d_m, d_m), w_k \rangle ds = \langle v_0, w_k \rangle \\ + \int_0^t \langle f_1(s, v_m), w_k \rangle ds + \sum_{j=1}^m \int_0^t \langle f_2(s, v_m) e_j, w_k \rangle d\beta_s^j, \\ \langle d_m, \beta_k \rangle + \int_0^t \langle B(v_m, d_m) - B(d_m, v_m) + A_1 d_m, \beta_k \rangle ds = \langle d_0, \beta_k \rangle, \\ \langle \Psi_m(t), A_\gamma \psi_k \rangle + \int_0^t \langle \mu_m + B_1(v_m, \Psi_m), A_\gamma \psi_k \rangle ds = \langle \Psi_0, \psi_k \rangle, \\ \mu_m = A_\gamma \Psi_m + f_\gamma(\Psi_m), \end{cases} \quad (63)$$

$$1 \leq k \leq m.$$

As it is the case with the proof of Theorem 1.2.1 of [10], we can get the existence and uniqueness of a solution $(v_m, d_m, \Psi_m) \in M_{\mathcal{F}_t}^2(0, T; \mathbb{V}_m)$ of (63) with continuous trajectories.

Step 2: Some estimates for the approximating sequence.

We have

$$\begin{aligned} & \mathcal{E}(v_m, d_m, \Psi_m) + 2 \int_0^t \|v_m(s)\|^2 + \|d_m(s)\|^2 + |\mu_m(s)|_{L^2}^2 ds = \mathcal{E}(v_0, d_0, \Psi_0) + 2 \int_0^t \langle f_1(s, v_m(s)), v_m(s) \rangle ds \\ & + 2 \int_0^t \langle f_2(s, v_m(s)), v_m(s) dW_s \rangle + \int_0^t \|f_2(s, v_m(s))\|_{\mathcal{L}^2(\mathbb{K}, H_1)}^2 ds, \quad \forall t \in [0, T]. \end{aligned} \quad (64)$$

This expression is equivalent to

$$\begin{aligned} \mathcal{E}(v_m, d_m, \Psi_m) + 2 \int_0^t \|v_m(s)\|^2 + \|d_m(s)\|^2 + |\mu_m(s)|_{L^2}^2 ds &= \mathcal{E}(v_0, d_0, \Psi_0) + 2 \int_0^t \langle f_1(s, v_m(s)), v_m(s) \rangle ds \\ + 2 \sum_{j=1}^m \int_0^t \langle f_2(s, v_m) e_j, v_m \rangle d\beta_s^j + \int_0^t \|f_2(s, v_m(s))\|_{\mathcal{L}^2(\mathbb{K}; H_1)}^2 ds, \quad \forall t \in [0, T]. \end{aligned} \quad (65)$$

We also have

$$\|f_2(s, v_m)\|_{\mathcal{L}^2(\mathbb{K}; H_1)}^2 \leq L_2^2 \|v_m\|_{L^2}^2 + c \|f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{K}; H_1)}^2, \quad (66)$$

and

$$\begin{aligned} |2 \langle f_1(s, v_m), v_m \rangle| &\leq 2L_1 \|v_m\|_{L^2} \|v_m\| + 2 \|f_1(s, 0)\|_{V_1^*} \|v_m\| \leq \frac{1}{4} \|v_m\| (4L_1 \|v_m\|_{L^2}) + \frac{1}{4} \|v_m\| (4 \|f_1(s, 0)\|) \\ &\leq \frac{1}{2} \left(\frac{1}{4} \|v_m\| \right)^2 + \frac{1}{2} (4L_1 \|v_m\|_{L^2})^2 + \frac{1}{2} \left(\frac{1}{4} \|v_m\| \right)^2 + \frac{1}{2} (4 \|f_1(s, 0)\|)^2 \\ &\leq \frac{1}{8} \|v_m\|^2 + cL_1^2 \|v_m\|_{L^2}^2 + c \|f_1(s, 0)\|_{V_1^*}^2. \end{aligned} \quad (67)$$

Thus

$$\begin{aligned} \int_0^t \|f_2(s, v_m)\|_{\mathcal{L}^2(\mathbb{K}; H_1)}^2 ds + \int_0^t |2 \langle f_1(s, v_m), v_m \rangle| ds &\leq \int_0^t (cL_1^2 + L_2^2) \|v_m\|_{L^2}^2 ds \\ + \int_0^t c \|f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{K}; H_1)}^2 ds + \int_0^t c \|f_1(s, 0)\|_{V_1^*}^2 ds + \int_0^t \frac{1}{8} \|v_m\|^2 ds, \end{aligned} \quad (68)$$

and then

$$\begin{aligned} \int_0^t \|f_2(s, v_m)\|_{\mathcal{L}^2(\mathbb{K}; H_1)}^2 ds + \int_0^t |2 \langle f_1(s, v_m), v_m \rangle| ds &\leq \int_0^t c \|f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{K}; H_1)}^2 ds \\ + \int_0^t c \|f_1(s, 0)\|_{V_1^*}^2 ds + (cL_1^2 + L_2^2) \int_0^t \|v_m\|_{L^2}^2 + \|d_m\|_{L^2}^2 + \|\Psi_m\|_{L^2}^2 + 2 \langle F_\gamma(\Psi), 1 \rangle ds. \end{aligned} \quad (69)$$

Furthermore, we deduce from (65)-(69) that

$$\begin{aligned} \mathcal{E}(v_m, d_m, \Psi_m)(t) + \int_0^t (\|v_m(s)\|^2 + \|d_m(s)\|^2 + |\mu_m(s)|_{L^2}^2) ds &\leq \mathcal{E}(v_0, d_0, \Psi_0) \\ + (cL_1^2 + L_2^2) \int_0^t \mathcal{E}(v_m, d_m, \Psi_m)(s) ds + c \int_0^t [\|f_1(s, 0)\|_{V_1^*}^2 + \|f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{K}; H_1)}^2] ds \\ + 2 \sum_{j=1}^m \int_0^t \langle f_2(s, v_m(s)) e_j, v_m(s) \rangle d\beta_s^j. \end{aligned} \quad (70)$$

For all $n \geq 1$, we consider the $\mathcal{F}_t$ -stopping time τ_n^m given by:

$$\tau_n^m = \min \left(T, \inf \left\{ t \in [0, T]; \mathcal{E}(v_m, d_m, \Psi_m)(t) + \int_0^t (\|v_m(s)\|^2 + \|d_m(s)\|^2 + |\mu_m(s)|_{L^2}^2) ds \geq n^2 \right\} \right). \quad (71)$$

With m fixed, the subsequence $\{\tau_n^m; n \geq 1\}$ is increasing up to T . From (70), we can write

$$\begin{aligned} \sup_{s \in [0, t \wedge \tau_n^m]} [\mathcal{E}(v_m, d_m, \Psi_m)(s)]^2 + c \left[\int_0^{t \wedge \tau_n^m} (\|v_m(s)\|^2 + \|d_m(s)\|^2 + |\mu_m(s)|_{L^2}^2) ds \right]^2 &\leq [\mathcal{E}(v_0, d_0, \Psi_0)]^2 \\ + c \int_0^t \sup_{r \in [0, s \wedge \tau_n^m]} [\mathcal{E}(v_m, d_m, \Psi_m)(r)]^2 ds + c \left[\int_0^T [\|f_1(s, 0)\|_{V_1^*}^2 + \|f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{K}; H_1)}^2] ds \right]^2 \\ + c \sup_{s \in [0, t \wedge \tau_n^m]} \left[\sum_{j=1}^m \int_0^s \langle f_2(r, v_m(s)) e_j, v_m(r) \rangle d\beta_r^j \right]^2. \end{aligned} \quad (72)$$

Using expectation and applying Doob's inequality, we get

$$\begin{aligned}
& \mathbb{E} \sup_{s \in [0, t \wedge \tau_n^m]} \left[\sum_{j=1}^m \int_0^s \langle f_2(r, v_m(s)) e_j, v_m(r) \rangle d\beta_r^j \right]^2 \leq c \mathbb{E} \sum_{j=1}^m \int_0^{t \wedge \tau_n^m} [\langle f_2(r, v_m(s)) e_j, v_m(s) \rangle]^2 ds \\
& \leq c \mathbb{E} \left(\sup_{s \in [0, t \wedge \tau_n^m]} \|v_m(s)\|_{L^2}^2 \int_0^{t \wedge \tau_n^m} \|f_2(s, v_m(s))\|_{\mathcal{L}^2(\mathbb{K}; H_1)}^2 ds \right) \\
& \leq \frac{1}{2} \mathbb{E} \sup_{r \in [0, t \wedge \tau_n^m]} [\mathcal{E}(v_m, d_m, \Psi_m)(s)]^2 + c \mathbb{E} \int_0^{t \wedge \tau_n^m} [\mathcal{E}(v_m, d_m, \Psi_m)(s)]^2 ds + c \mathbb{E} \left[\int_0^{t \wedge \tau_n^m} \|f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{K}; H_1)}^2 ds \right]^2.
\end{aligned} \tag{73}$$

It comes from (73) and (72) that

$$\begin{aligned}
& \mathbb{E} \left(\sup_{s \in [0, T]} [\mathcal{E}(v_m, d_m, \Psi_m)(s)]^2 \right) + \mathbb{E} \left[\int_0^{t \wedge \tau_n^m} (\|v_m(s)\|^2 + \|d_m(s)\|^2 + |\mu_m(s)|_{L^2}^2) ds \right]^2 \leq c \mathbb{E} [\mathcal{E}(v_0, d_0, \Psi_0)]^2 \\
& + c \mathbb{E} \left[\int_0^T \|f_1(s, 0)\|_{V_1^*}^2 ds \right]^2 + c \mathbb{E} \left[\int_0^T \|f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{K}, H_1)}^2 ds \right]^2 \\
& + 2c \mathbb{E} \left[\int_0^T \|f_1(s, 0)\|_{V_1^*}^2 ds \right] \left[\int_0^T \|f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{K}, H_1)}^2 ds \right] + \frac{1}{2} \mathbb{E} \sup_{r \in [0, t \wedge \tau_n^m]} [\mathcal{E}(v_m, d_m, \Psi_m)(s)]^2 \\
& + c \mathbb{E} \int_0^{t \wedge \tau_n^m} [\mathcal{E}(v_m, d_m, \Psi_m)(s)]^2 ds + c \mathbb{E} \left[\int_0^{t \wedge \tau_n^m} \|f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{K}; H_1)}^2 ds \right]^2.
\end{aligned} \tag{74}$$

However $\tau_n^m \uparrow T$ when n tends to ∞ . Applying Gronwall's lemma, we get

$$\begin{aligned}
& \mathbb{E} \left(\sup_{s \in [0, T]} [\mathcal{E}(v_m, d_m, \Psi_m)(s)]^2 \right) + \mathbb{E} \left[\int_0^T (\|v_m(s)\|^2 + \|d_m(s)\|^2 + |\mu_m(s)|_{L^2}^2) ds \right]^2 \leq c \mathbb{E} [\mathcal{E}(v_0, d_0, \Psi_0)]^2 \\
& + c \mathbb{E} \left[\int_0^T \|f_1(s, 0)\|_{V_1^*}^2 ds \right]^2 + c \mathbb{E} \left[\int_0^T \|f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{K}, H_1)}^2 ds \right]^2.
\end{aligned} \tag{75}$$

In addition, we have

$$\begin{aligned}
& |B(v_m, v_m)|_{V_1^*} \leq c \|v_m\|_{L^2} \|v_m\|, \\
& |B(d_m, d_m)|_{V_1^*} \leq c \|d_m\|_{L^2} \|d_m\|, \\
& |B(v_m, d_m)|_{V_2^*} \leq c \|v_m\|_{L^2} \|d_m\|, \\
& |B(d_m, v_m)|_{V_2^*} \leq c \|d_m\|_{L^2} \|v_m\|, \\
& |R_0(A_\gamma \Psi_m, \Psi_m)|_{V_1^*} = |R_1(\Psi_m, \Psi_m)|_{V_1^*} \leq c \|\Psi_m\| \|A_\gamma \Psi_m\|_{L^2}, \\
& |B_1(v_m, \Psi_m)|_{L^2} \leq c \|v_m\|_{L^2}^{1/2} \|v_m\|^{1/2} \|\Psi_m\|^{1/2} \|A_\gamma \Psi_m\|_{L^2}^{1/2}.
\end{aligned} \tag{76}$$

From (63)₄, we can note that

$$A_\gamma \Psi_m = \mu_m - f_\gamma(\Psi_m), \quad \text{and} \quad |A_\gamma \Psi_m|_{L^2} = |\mu_m - f_\gamma(\Psi_m)| \leq c \|\mu_m\|_{L^2} + c \|f_\gamma(\Psi_m)\|_{L^2}. \tag{77}$$

Taking into account (8), we also write

$$|f_\gamma(\Psi_m)|_{L^2} \leq c(1 + \|\Psi_m\|^4) \leq c(1 + (\mathcal{E}(v_0, d_0, \Psi_0))^2). \tag{78}$$

It comes from (75)-(78) that

$$\begin{aligned}
& \mathbb{E} \sup_{s \in [0, T]} |\mathcal{E}(v_m, d_m, \Psi_m)(s)|_{\mathbb{Y}}^2 \leq C, \quad \mathbb{E} \int_0^T \|(v_m(s), d_m(s), \Psi_m(s))\|_{\mathbb{V}}^2 ds \leq C, \\
& \mathbb{E} \int_0^T |A_\gamma \Psi_m(s)|_{L^2} ds \leq C, \quad \mathbb{E} \sup_{t \in [0, T]} |f_\gamma(\Psi_m)|_{L^2} \leq C, \quad \mathbb{E} \int_0^T |B(v_m, d_m)|_{V_2^*}^2 ds \leq C,
\end{aligned} \tag{79}$$

$$\begin{aligned}
& \mathbb{E} \int_0^T \left[|B(v_m, v_m)|_{V_1^*}^2 + |R_0(A_\gamma \Psi_m, \Psi_m)|_{V_1^*}^2 + |B_1(v_m, \Psi_m)|_{L^2}^2 \right] ds \leq C, \\
& \mathbb{E} \int_0^T \left[|B(d_m, d_m)|_{V_1^*}^2 + |B(d_m, v_m)|_{V_2^*}^2 \right] ds \leq C, \quad \mathbb{E} \sup_{s \in [0, T]} |\mathcal{E}(v_m, d_m, \Psi_m)(s)|_{\mathbb{Y}}^4 \leq C, \\
& \mathbb{E} \left[\int_0^T \| (v_m(s), d_m(s), \Psi_m(s)) \|_{\mathbb{V}}^2 ds \right]^2 \leq C.
\end{aligned} \tag{80}$$

Step 3: Taking the limit. Taking into account (79) and (80), we can find a subsequence still denoted $\{(v_m, d_m, \Psi_m)\}$ such as

$$\begin{aligned}
& (v_m, d_m, \Psi_m) \rightharpoonup (v, d, \Psi) \in M_{\mathcal{F}_t}^2(0, T, \mathbb{V}), \\
& (v_m(0), d_m(0), \Psi_m(0)) \rightharpoonup (\xi_1, \xi_2, \xi_3) \in L^4(\Omega, \mathcal{F}_0, \mathbb{P}, \mathbb{Y}) \\
& B(v_m, v_m) \rightharpoonup \beta_0^b \in M_{\mathcal{F}_t}^2(0, T, V_1^*), \\
& B(d_m, d_m) \rightharpoonup \beta_2^b \in M_{\mathcal{F}_t}^2(0, T, V_1^*), \\
& R_0(A_\gamma \Psi_m, \Psi_m) \rightharpoonup r_0^b \in M_{\mathcal{F}_t}^2(0, T, V_1^*), \\
& B_1(v_m, \Psi_m) \rightharpoonup \beta_1^b \in M_{\mathcal{F}_t}^2(0, T, H_3), \\
& B(v_m, d_m) \rightharpoonup \beta_3^b \in M_{\mathcal{F}_t}^2(0, T, V_2^*), \\
& B(d_m, v_m) \rightharpoonup \beta_4^b \in M_{\mathcal{F}_t}^2(0, T, V_2^*), \\
& f_1(t, v_m) \rightharpoonup h_1^b \in M_{\mathcal{F}_t}^2(0, T, V_1^*), \\
& f_2(t, v_m) \rightharpoonup h_2^b \in M_{\mathcal{F}_t}^2(0, T, \mathcal{L}^2(\mathbb{K}, H_1)), \\
& f_\gamma(\Psi_m) \rightharpoonup f^b \in M_{\mathcal{F}_t}^2(0, T, \mathcal{L}^2(\mathbb{K}, H_3)).
\end{aligned} \tag{81}$$

As in [41], we can check that $(v, d, \Psi) \in L^2(\Omega, C(0, T; \mathbb{Y}))$ and verifies for all $0 \leq t \leq T$,

$$\begin{aligned}
& v(t) + \int_0^t A_0 v ds + \int_0^t \beta_0^b(s) ds = v_0 + \int_0^t \beta_2^b(s) ds + \int_0^t (r_0^b(s) + h_1^b(s)) ds + \int_0^t h_2^b(s) dW_s, \\
& d(t) + \int_0^t \beta_3^b(s) - \beta_4^b(s) ds + \int_0^t A_1(d) ds = b_0, \\
& \Psi(t) + \int_0^t \mu^b ds + \int_0^t \beta_1^b(s) ds = \Psi_0, \\
& \mu^b = A_\gamma \Psi + f^b.
\end{aligned} \tag{82}$$

We now show that

$$\beta_0^b = B(v, v), \quad r_0^b = R_0(A_\gamma \Psi, \Psi), \quad \beta_1^b = B_1(v, \Psi), \quad \beta_2^b = B(d, d), \quad \beta_3^b = B(v, d), \quad \beta_4^b = B(d, v), \quad f^b = f_\gamma(\Psi).$$

Set

$$(\tilde{v}_m, \tilde{d}_m, \tilde{\Psi}_m) = \Pi_m(v, d, \Psi),$$

where $\Pi_m = (\Pi_m^1, \Pi_m^2, \Pi_m^3)$ is the orthogonal projection of $\mathbb{Y}$ onto $\mathbb{Y}_m$. It comes that

$$\begin{aligned}
& |(\tilde{v}_m, \tilde{d}_m, \tilde{\Psi}_m)|_{\mathbb{Y}} \leq |(v, d, \Psi)|_{\mathbb{Y}}, \\
& \|(\tilde{v}_m, \tilde{d}_m, \tilde{\Psi}_m)\|_{\mathbb{V}} \leq c \|(v, d, \Psi)\|_{\mathbb{V}}, \\
& (\tilde{v}_m, \tilde{d}_m, \tilde{\Psi}_m) \longrightarrow (v, d, \Psi) \in M_{\mathcal{F}_t}^2(0, T, \mathbb{V}).
\end{aligned} \tag{83}$$

From (63) and (82), we deduce that

$$\left\{ \begin{aligned} & \langle \tilde{v}_m(t) - v_m(t), w_k \rangle + \int_0^t \langle A_0(\tilde{v}_m - v_m), w_k \rangle ds + \int_0^t \langle \beta_0^b - B(v_m, v_m), w_k \rangle ds - \int_0^t \langle \beta_2^b - B(d_m, d_m), w_k \rangle ds \\ &= \int_0^t \langle r_0^b - R_0(A_\gamma \Psi_m, \Psi_m), w_k \rangle ds + \int_0^t \langle h_1^b - f_1(s, v_m), w_k \rangle ds + \sum_{j=1}^m \int_0^t \langle h_2^b e_j - f_2(s, v_m) e_j, w_k \rangle d\beta_s^j \\ &+ \sum_{j=m+1}^\infty \int_0^t \langle h_2^b(s) e_j, w_k \rangle d\beta_s^j, \\ & \langle \tilde{d}_m(t) - d_m(t), \beta_k \rangle + \int_0^t \langle \beta_3^b - B(v_m, d_m), \beta_k \rangle ds - \int_0^t \langle \beta_4^b - B(d_m, v_m), \beta_k \rangle ds \\ &+ \int_0^t \langle A_1(\tilde{d}_m - d_m), \beta_k \rangle ds = 0, \\ & \langle \tilde{\Psi}_m(t) - \Psi_m(t), A_\gamma \psi_k \rangle + \int_0^t \langle \tilde{\mu}_m - \mu_m, A_\gamma \psi_k \rangle ds + \int_0^t \langle \beta_1^b - B_1(v_m, \Psi_m), A_\gamma \psi_k \rangle ds = 0, \\ & \langle \tilde{\mu}_m - \mu_m, A_\gamma \psi_k \rangle = \langle A_\gamma(\tilde{\Psi}_m(t) - \Psi_m(t)), A_\gamma \psi_k \rangle + \langle f^b - f_\gamma(\Psi_m), A_\gamma \psi_k \rangle, \quad 1 \leq k \leq m. \end{aligned} \right. \quad (84)$$

Note that

$$\begin{aligned} \beta_0^b - B(v_m, v_m) &= \beta_0^b - B(\tilde{v}_m, \tilde{v}_m) + B(\tilde{v}_m - v_m, \tilde{v}_m) + B(v_m, \tilde{v}_m - v_m), \\ \beta_2^b - B(d_m, d_m) &= \beta_2^b - B(\tilde{d}_m, \tilde{d}_m) + B(\tilde{d}_m - d_m, \tilde{d}_m) + B(d_m, \tilde{d}_m - d_m), \\ r_0^b - R_0(A_\gamma \Psi_m, \Psi_m) &= r_0^b - R_0(A_\gamma \tilde{\Psi}_m, \tilde{\Psi}_m) + R_0(A_\gamma(\tilde{\Psi}_m - \Psi_m), \tilde{\Psi}_m) + R_0(A_\gamma \Psi_m, \tilde{\Psi}_m - \Psi_m), \\ \beta_1^b - B_1(v_m, \Psi_m) &= \beta_1^b - B_1(\tilde{v}_m, \tilde{\Psi}_m) + B_1(\tilde{v}_m - v_m, \tilde{\Psi}_m) + B_1(v_m, \tilde{\Psi}_m - \Psi_m), \\ \beta_3^b - B(v_m, d_m) &= \beta_3^b - B(\tilde{v}_m, \tilde{d}_m) + B(\tilde{v}_m - v_m, \tilde{d}_m) + B(v_m, \tilde{d}_m - d_m), \\ \beta_4^b - B(d_m, v_m) &= \beta_4^b - B(\tilde{d}_m, \tilde{v}_m) + B(\tilde{d}_m - d_m, \tilde{v}_m) + B(d_m, \tilde{v}_m - v_m), \\ f^b - f_\gamma(\Psi_m) &= f^b - f_\gamma(\tilde{\Psi}_m) + f_\gamma(\tilde{\Psi}_m) - f_\gamma(\Psi_m). \end{aligned} \quad (85)$$

Set $\theta_m = \tilde{v}_m - v_m$, $\rho_m = \tilde{\Psi}_m - \Psi_m$, $\alpha_m = \tilde{d}_m - d_m$.

From the Itô formula, we have

$$d\langle \theta_m, w_k \rangle^2 = 2\langle \theta_m, w_k \rangle d\langle \theta_m, w_k \rangle + \sum_{j=1}^m [\langle h_1^b e_j - f_1(t, v_m) e_j, w_k \rangle]^2 dt + \sum_{j=m+1}^\infty [\langle h_1^b e_j, w_k \rangle]^2 dt.$$

Thus

$$\begin{aligned} & |\theta_m(t)|_{L^2}^2 + 2 \int_0^t (|\theta_m(t)|^2 + \langle \beta_0^b - B(v_m, v_m), \theta_m \rangle) ds = 2 \int_0^t \langle r_0^b - R_0(A_\gamma \Psi_m, \Psi_m), \theta_m \rangle ds \\ &+ 2 \int_0^t \langle \beta_2^b - B(d_m, d_m), \theta_m \rangle ds + 2 \int_0^t \langle h_1^b - f_1(s, v_m), \theta_m \rangle ds + 2 \sum_{j=1}^m \int_0^t \langle f_2 e_j - f_2(s, v_m) e_j, \theta_m \rangle d\beta_s^j \\ &+ 2 \sum_{j=m+1}^\infty \int_0^t \langle h_2^b(s) e_j, \theta_m \rangle d\beta_s^j + \sum_{j=1}^m \int_0^t |\Pi_m^1(f_2 e_j - f_2(s, v_m) e_j)|_{L^2}^2 ds + \sum_{j=m+1}^\infty \int_0^t |\Pi_m^1 f_2 e_j|_{L^2}^2 ds. \end{aligned} \quad (86)$$

Now, we take the duality of (84)₂ with α_m and we obtain

$$\frac{d}{dt} \|\alpha_m\|^2 + 2\langle \beta_3^b - B(v_m, d_m), \alpha_m \rangle - 2\langle \beta_4^b - B(d_m, v_m), \alpha_m \rangle + 2\langle A_1 d_m, \alpha_m \rangle = 0. \quad (87)$$

Changing ψ_k in (84)₃ by ρ_m , we have

$$\begin{aligned} & \frac{d}{dt} \|\rho_m\|^2 + 2|A_\gamma \rho_m|_{L^2}^2 + 2\langle \beta_1^b - B_1(\tilde{v}_m, \tilde{\Psi}_m), A_\gamma \rho_m \rangle + 2\langle B_1(\theta_m, \tilde{\Psi}_m), A_\gamma \rho_m \rangle \\ &+ 2\langle B_1(v_m, \rho_m), A_\gamma \rho_m \rangle + \langle f^b - f_\gamma(\Psi_m), A_\gamma \rho_m \rangle = 0. \end{aligned} \quad (88)$$

We also have the inequalities below:

$$\begin{aligned} \langle \beta_0^b - B(v_m, v_m), \theta_m \rangle &= \langle \beta_0^b - B(\tilde{v}_m, \tilde{v}_m), \theta_m \rangle + \langle B(\theta_m, \tilde{v}_m), \theta_m \rangle \\ &\leq \langle \beta_0^b - B(\tilde{v}_m, \tilde{v}_m), \theta_m \rangle + \frac{1}{4} \|\theta_m\|^2 + c \|\tilde{v}_m\|^2 \|\theta_m\|_{L^2}^2, \end{aligned} \quad (89)$$

and

$$\begin{aligned} \beta_2^b - B(d_m, d_m) &= \beta_2^b - B(\tilde{d}_m, \tilde{d}_m) + B(\tilde{d}_m - d_m, \tilde{d}_m) + B(d_m, \tilde{d}_m - d_m), \\ \langle \beta_2^b - B(d_m, d_m), \theta_m \rangle &= \langle \beta_2^b - B(\tilde{d}_m, \tilde{d}_m), \theta_m \rangle + \langle B(\alpha_m, \tilde{d}_m) + B(d_m, \alpha_m), \theta_m \rangle \\ &\leq \langle \beta_2^b - B(\tilde{d}_m, \tilde{d}_m), \theta_m \rangle + \frac{1}{4} \|\alpha_m\|^2 + c \|\tilde{d}_m\|^2 \|\theta_m\|_{L^2}^2 + \frac{1}{4} \|d_m\|^2 + c \|\alpha_m\|^2 \|\theta_m\|_{L^2}^2, \end{aligned} \quad (90)$$

$$\begin{aligned} \langle \beta_1^b - B_1(v_m, \Psi_m), A_\gamma \rho_m \rangle &= \langle \beta_1^b - B_1(\tilde{v}_m, \tilde{\Psi}_m), A_\gamma \rho_m \rangle + \langle B_1(\theta_m, \tilde{\Psi}_m), A_\gamma \rho_m \rangle + \langle B_1(v_m, \rho_m), A_\gamma \rho_m \rangle \\ &\leq \langle \beta_1^b - B_1(\tilde{v}_m, \tilde{\Psi}_m), A_\gamma \rho_m \rangle + \frac{1}{4} (\|\theta_m\|^2 + \|A_\gamma \rho_m\|_{L^2}^2) + c \|\Psi_m\|^2 \|A_\gamma \Psi_m\|_{L^2}^2 \|\theta_m\|_{L^2}^2 + c \|\tilde{v}_m\|_{L^2}^2 \|\tilde{\Psi}_m\|^2 \|\rho_m\|^2, \end{aligned} \quad (91)$$

$$\begin{aligned} \langle \beta_3^b - B(v_m, d_m), \alpha_m \rangle &= \langle \beta_3^b - B(\tilde{v}_m, \tilde{d}_m), \alpha_m \rangle + \langle B(\theta_m, \tilde{d}_m), \alpha_m \rangle + \langle B(v_m, \alpha_m), \alpha_m \rangle \\ &\leq \langle \beta_3^b - B(\tilde{v}_m, \tilde{d}_m), \alpha_m \rangle + \frac{1}{4} \|\theta_m\|^2 + c \|\tilde{d}_m\|^2 \|\alpha_m\|_{L^2}^2, \end{aligned} \quad (92)$$

$$\begin{aligned} \langle \beta_4^b - B(d_m, v_m), \alpha_m \rangle &= \langle \beta_4^b - B(\tilde{d}_m, \tilde{v}_m), \alpha_m \rangle + \langle B(\alpha_m, \tilde{v}_m), \alpha_m \rangle + \langle B(d_m, \theta_m), \alpha_m \rangle \\ &\leq \langle \beta_4^b - B(\tilde{d}_m, \tilde{v}_m), \alpha_m \rangle + \frac{1}{4} \|\alpha_m\|^2 + c \|\tilde{v}_m\|^2 \|\alpha_m\|_{L^2}^2 + \frac{1}{4} \|d_m\|^2 + c \|\theta_m\|^2 \|\alpha_m\|_{L^2}^2, \end{aligned} \quad (93)$$

$$\begin{aligned} \langle r_0^b - R_0(A_\gamma \Psi_m, \Psi_m), \theta_m \rangle &= \langle r_0^b - R_0(A_\gamma \tilde{\Psi}_m, \tilde{\Psi}_m), \theta_m \rangle + \langle R_0(A_\gamma \rho_m, \tilde{\Psi}_m), \theta_m \rangle + \langle R_0(A_\gamma \Psi_m, \rho_m), \theta_m \rangle \\ &\leq \langle r_0^b - R_0(A_\gamma \tilde{\Psi}_m, \tilde{\Psi}_m), \theta_m \rangle + \frac{1}{4} (\|\theta_m\|^2 + \|A_\gamma \rho_m\|_{L^2}^2) + c \|A_\gamma \Psi_m\|_{L^2}^2 (\|\theta_m\|_{L^2}^2 + \|\rho_m\|^2) \\ &\quad + c \|\tilde{\Psi}_m\|^2 \|A_\gamma \tilde{\Psi}_m\|_{L^2}^2 \|\theta_m\|_{L^2}^2, \end{aligned} \quad (94)$$

$$\begin{aligned} \langle f^b - f_\gamma(\Psi_m), A_\gamma \rho_m \rangle &= \langle f^b - f_\gamma(\tilde{\Psi}_m), A_\gamma \rho_m \rangle + \langle f_\gamma(\tilde{\Psi}_m) - f_\gamma(\Psi_m), A_\gamma \rho_m \rangle \\ &\leq \langle f^b - f_\gamma(\tilde{\Psi}_m), A_\gamma \rho_m \rangle + \frac{1}{8} \|A_\gamma \rho_m\|_{L^2}^2 + Q_1(\|\tilde{\Psi}_m\|, \|\Psi_m\|) \|\rho_m\|^2, \end{aligned} \quad (95)$$

$$\begin{aligned} \sum_{j=1}^m |\Pi_m^1(h_2^b e_j - f_2(s, v_m) e_j)|_{L^2}^2 &\leq \|h_2^b - f_2(s, v_m)\|_{\mathcal{L}^2(\mathbb{K}, H_1)}^2 \\ &\leq \|f_2(s, v) - f_2(s, v_m)\|_{\mathcal{L}^2(\mathbb{K}, H_1)}^2 + 2 \langle h_2^b - f_2(s, v_m), h_2^b - f_2(s, v_m) \rangle - \|f_2(s, v) - f_2(s, v_m)\|_{\mathcal{L}^2(\mathbb{K}, H_1)}^2 \\ &\leq 2L_2 \|v - \tilde{v}_m\|_{L^2}^2 + 2L_2 \|v_m - \tilde{v}_m\|_{L^2}^2 + 2 \langle h_2^b - f_2(s, v_m), h_2^b - f_2(s, v_m) \rangle - \|f_2(s, v) - f_2(s, v_m)\|_{\mathcal{L}^2(\mathbb{K}, H_1)}^2, \end{aligned} \quad (96)$$

$$\langle f_1(s, \tilde{v}_m) - f_1(s, v_m), \theta_m \rangle \leq L_1 \|\theta_m\|_{L^2} \|\theta_m\| \leq \frac{1}{4} \|\theta_m\|^2 + cL_1^2 \|\theta_m\|_{L^2}^2. \quad (97)$$

Set

$$\begin{aligned} Z(t) &= \|\theta_m(t)\|_{L^2}^2 + \|\alpha_m(t)\|^2 + \|\rho_m(t)\|^2 = \|(\tilde{v}_m - v_m)(t)\|_{L^2}^2 + \|(\tilde{d}_m - d_m)(t)\|^2 + \|(\tilde{\Psi}_m - \Psi_m)(t)\|^2, \\ Y_1(t) &= c \|\tilde{v}_m\|^2 + c \|\tilde{d}_m\|^2 + c \|\Psi_m\|^2 \|A_\gamma \Psi_m\|_{L^2}^2 + c \|\tilde{v}_m\|_{L^2}^2 \|\tilde{v}_m\|^2 + c \|A_\gamma \Psi_m\|_{L^2}^2 \\ &\quad + c \|\tilde{\Psi}_m\|^2 \|A_\gamma \tilde{\Psi}_m\|_{L^2}^2 + Q_1(\|\tilde{\Psi}_m\|, \|\Psi_m\|) + cL_1^2 + cL_2^2, \\ K_2(t) &= \|\theta_m\|^2 + \|\alpha_m\|^2 + \|A_\gamma \rho_m\|_{L^2}^2. \end{aligned}$$

Let

$$\sigma(t) = \exp \left(- \int_0^t Y_1(s) ds \right).$$

Using (86)-(97), it comes from Itô's formula that

$$\begin{aligned}
& \sigma(t)Z(t) + \int_0^t \sigma(s)K_2(s)ds + \int_0^t \sigma(s)||h_2^b - f_2(s, v)||_{\mathcal{L}^2(\mathbb{K}, H_1)}^2 ds \leq c \int_0^t \sigma(s)\langle -\beta_0^b + B(\tilde{v}_m, \tilde{v}_m), \theta_m \rangle ds \\
& + c \int_0^t \sigma(s)\langle \beta_2^b - B(\tilde{d}_m, \tilde{d}_m), \theta_m \rangle ds + c \int_0^t \sigma(s)\langle -\beta_1^b + B_1(\tilde{v}_m, \tilde{\Psi}_m), A_\gamma \rho_m \rangle ds \\
& + c \int_0^t \sigma(s)\langle -\beta_3^b + B(\tilde{v}_m, \tilde{d}_m), \alpha_m \rangle ds + c \int_0^t \sigma(s)\langle \beta_4^b - B(\tilde{d}_m, \tilde{v}_m), \alpha_m \rangle ds \\
& + c \int_0^t \sigma(s)\langle r_0^b - R_0(A_\gamma \tilde{\Psi}_m, \tilde{\Psi}_m), \theta_m \rangle ds + c \int_0^t \sigma(s)\langle h_1^b - f_1(s, \tilde{v}_m), \theta_m \rangle ds \\
& + 2L_2^2 \int_0^t \sigma(s)|v(s) - \tilde{v}_m(s)|_{L^2}^2 ds + \sum_{j=m+1}^{\infty} \int_0^t \sigma(s)|\Pi_m^1 f_2 e_j|_{L^2}^2 ds \\
& + 2 \int_0^t \sigma(s)\langle h_2^b - f_2(s, v_m), h_2^b - f_2(s, v) \rangle ds + 2 \sum_{j=1}^m \int_0^t \sigma(s)\langle h_2^b(s) e_j - f_2(s, v_m) e_j, \theta_m \rangle d\beta_s^j \\
& + \sum_{j=m+1}^{\infty} \int_0^t \sigma(s)\langle h_2^b(s) e_j, \theta_m \rangle d\beta_s^j.
\end{aligned} \tag{98}$$

For all $n \geq 1$, we consider the $\mathcal{F}_t$ -stopping time τ_n given by:

$$\tau_n = \min \left(T, \inf \left\{ t \in [0, T]; |v(t, d, \Psi)|_{\mathbb{V}}^2 + \int_0^t ||(v, d, \Psi)||_{\mathbb{V}}^2 ds \geq n^2 \right\} \right). \tag{99}$$

We deduce from (98) that

$$\begin{aligned}
& \mathbb{E}\sigma(\tau_n)Z(\tau_n) + c\mathbb{E} \int_0^{\tau_n} \sigma(s)K_2(s)ds + \mathbb{E} \int_0^{\tau_n} \sigma(s)||h_2^b - f_2(s, v)||_{\mathcal{L}^2(\mathbb{K}, H_1)}^2 ds \\
& \leq \sum_{j=m+1}^{\infty} \mathbb{E} \int_0^T \sigma(s)|f_2 e_j|_{L^2}^2 ds + 2L_2^2 \mathbb{E} \int_0^T \sigma(s)|v(s) - \tilde{v}_m(s)|_{L^2}^2 ds + 2\mathbb{E} \int_0^t \sigma(s)\langle \beta_4^b - B(\tilde{d}_m, \tilde{v}_m), \alpha_m \rangle ds \\
& + 2\mathbb{E} \int_0^{\tau_n} \sigma(s)\langle -\beta_0^b + B(\tilde{v}_m, \tilde{v}_m) + h_1^b - f_1(s, \tilde{v}_m) + r_0^b - R_0(A_\gamma \tilde{\Psi}_m, \tilde{\Psi}_m) + \beta_2^b - B(\tilde{d}_m, \tilde{d}_m), \theta_m \rangle ds \\
& + 2\mathbb{E} \int_0^{\tau_n} \sigma(s)\langle -\beta_1^b + B_1(\tilde{v}_m, \tilde{\Psi}_m), A_\gamma \rho_m \rangle ds + 2\mathbb{E} \int_0^t \sigma(s)\langle -\beta_3^b + B(\tilde{v}_m, \tilde{d}_m), \alpha_m \rangle ds \\
& + 2\mathbb{E} \int_0^{\tau_n} \sigma(s)\langle h_2^b - f_2(s, v_m), h_2^b - f_2(s, v) \rangle ds.
\end{aligned} \tag{100}$$

Claim 1. The right side of (100) goes to 0 as m goes to $+\infty$.

(i). Firstly, we note that

$$\lim_{m \rightarrow \infty} \left(\sum_{j=m+1}^{\infty} \mathbb{E} \int_0^T \sigma(s)|f_2 e_j|_{L^2}^2 ds + \mathbb{E} \int_0^T \sigma(s)|v(s) - \tilde{v}_m(s)|_{L^2}^2 ds \right) = 0. \tag{101}$$

Furthermore, since

$$f_2(t, v_m) \rightharpoonup h_2^b(t) \quad \text{in} \quad M_{\mathcal{F}_t}^2(0, T, \mathcal{L}^2(\mathbb{K}, H_1)),$$

and

$$1_{[0, \tau_n]} \sigma(t)(h_2^b(t) - f_2(t, v)) \in M_{\mathcal{F}_t}^2(0, T, \mathcal{L}^2(\mathbb{K}, H_1)),$$

we deduce that

$$\lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s)\langle h_2^b - f_2(s, \tilde{v}_m), \theta_m \rangle ds = 0. \tag{102}$$

(ii). Let us now show that

$$\lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s)\langle -\beta_0^b + B(\tilde{v}_m, \tilde{v}_m), \theta_m \rangle ds = 0. \tag{103}$$

Note that

$$\begin{aligned} (v_m, d_m, \Psi_m) &\rightharpoonup (v, d, \Psi), \quad (\tilde{v}_m, \tilde{d}_m, \tilde{\Psi}_m) \rightharpoonup (v, d, \Psi), \\ (\tilde{v}_m, \tilde{d}_m, \tilde{\Psi}_m) - (v_m, d_m, \Psi_m) &\rightharpoonup (0, 0, 0) \quad \text{in } M_{\mathcal{F}_t}^2(0, T, \mathbb{V}). \end{aligned} \quad (104)$$

We also have

$$\begin{aligned} \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle -\beta_0^b + B(\tilde{v}_m, \tilde{v}_m), \theta_m \rangle ds &= \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle -\beta_0^b + B(v, v), \theta_m \rangle ds \\ + \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle -B(v, v) + B(\tilde{v}_m, \tilde{v}_m), \theta_m \rangle ds. \end{aligned} \quad (105)$$

Due to the fact that $1_{[0, \tau_n]} \sigma(t) (-\beta_0^b + B(v, v)) \in M_{\mathcal{F}_t}^2(0, T, V_1^*)$ in view of (104) and (105), we get

$$\lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle -\beta_0^b + B(v, v), \theta_m \rangle ds = 0. \quad (106)$$

We also write that

$$\| -B(v, v) + B(\tilde{v}_m, \tilde{v}_m) \|_{V_1^*} \leq c |\tilde{v}_m - v|_{L^2}^{1/2} \| \tilde{v}_m - v \|^{1/2} (\| \tilde{v}_m \|^{1/2} | \tilde{v}_m |_{L^2}^{1/2} + \| v \|^{1/2} | v |_{L^2}^{1/2}),$$

which leads to

$$\begin{aligned} \| 1_{[0, \tau_n]} (-B(v, v) + B(\tilde{v}_m, \tilde{v}_m)) \|_{V_1^*} &\rightarrow 0 \quad \text{as } m \rightarrow \infty, \quad dt \times d\mathbb{P} - a.e., \\ \| 1_{[0, \tau_n]} (-B(v, v) + B(\tilde{v}_m, \tilde{v}_m)) \|_{V_1^*} &\leq cn \| v \| \in M_{\mathcal{F}_t}^2(0, T, \mathbb{R}). \end{aligned}$$

Therefore, we have

$$\lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle -B(v, v) + B(\tilde{v}_m, \tilde{v}_m), \theta_m \rangle ds = 0. \quad (107)$$

From (106) and (107), we finally get

$$\lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle -\beta_0^b + B(\tilde{v}_m, \tilde{v}_m), \theta_m \rangle ds = 0, \quad (108)$$

which proves (103).

(iii). We now prove that

$$\lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle r_0^b - R_0(A_\gamma \tilde{\Psi}_m, \tilde{\Psi}_m), \theta_m \rangle ds = 0. \quad (109)$$

We have

$$\begin{aligned} \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle r_0^b - R_0(A_\gamma \tilde{\Psi}_m, \tilde{\Psi}_m), \theta_m \rangle ds &= \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle r_0^b - R_0(A_\gamma \Psi, \Psi), \theta_m \rangle ds \\ + \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle R_0(A_\gamma \Psi, \Psi) - R_0(A_\gamma \tilde{\Psi}_m, \tilde{\Psi}_m), \theta_m \rangle ds. \end{aligned} \quad (110)$$

Since $1_{[0, \tau_n]} \sigma(t) (r_0^b - R_0(A_\gamma \Psi, \Psi)) \in M_{\mathcal{F}_t}^2(0, T, V_1^*)$, it follows from (104) and (110) that

$$\lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle r_0^b - R_0(A_\gamma \Psi, \Psi), \theta_m \rangle ds = 0. \quad (111)$$

In addition

$$\begin{aligned} \| R_0(A_\gamma \tilde{\Psi}_m, \tilde{\Psi}_m) - R_0(A_\gamma \Psi, \Psi) \|_{V_1^*} &= \| R_1(\tilde{\Psi}_m, \tilde{\Psi}_m) - R_1(\Psi, \Psi) \|_{V_1^*} \\ &\leq c |A_\gamma(\tilde{\Psi}_m - \Psi)|_{L^2}^{1/2} \| \tilde{\Psi}_m - \Psi \|^{1/2} \| \Psi \|^{1/2} |A_\gamma \Psi|_{L^2}^{1/2} + c \| \tilde{\Psi}_m - \Psi \|^{1/2} |A_\gamma(\tilde{\Psi}_m - \Psi)|_{L^2}^{1/2} |A_\gamma \tilde{\Psi}_m|_{L^2}^{1/2} \| \tilde{\Psi}_m \|^{1/2}, \end{aligned}$$

which implies that

$$\begin{aligned} \| 1_{[0, \tau_n]} (R_0(A_\gamma \tilde{\Psi}_m, \tilde{\Psi}_m) - R_0(A_\gamma \Psi, \Psi)) \|_{V_1^*} &\rightarrow 0 \quad \text{as } m \rightarrow \infty, \quad dt \times d\mathbb{P} - a.e., \\ \| 1_{[0, \tau_n]} (R_0(A_\gamma \tilde{\Psi}_m, \tilde{\Psi}_m) - R_0(A_\gamma \Psi, \Psi)) \|_{V_1^*} &\leq cn |A_\gamma \Psi|_{L^2} \quad \text{in } M_{\mathcal{F}_t}^2(0, T, \mathbb{R}). \end{aligned}$$

It comes that

$$\lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle R_0(A_\gamma \tilde{\Psi}_m, \tilde{\Psi}_m) - R_0(A_\gamma \Psi, \Psi), \theta_m \rangle ds = 0. \quad (112)$$

We can conclude from (111) and (112) that

$$\lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle r_0^b - R_0(A_\gamma \tilde{\Psi}_m, \tilde{\Psi}_m), \theta_m \rangle ds = 0, \quad (113)$$

which proves (109).

(iv). The same methods used in (105) and (107), allow to get

$$\begin{aligned} & \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle \beta_2^b - B(\tilde{d}_m, \tilde{d}_m), \theta_m \rangle ds = \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle \beta_2^b - B(d, d), \theta_m \rangle ds \\ & + \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle B(d, d) - B(\tilde{d}_m, \tilde{d}_m), \theta_m \rangle ds = 0, \\ & \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle -\beta_3^b + B(\tilde{v}_m, \tilde{d}_m), \alpha_m \rangle ds = \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle -\beta_3^b + B(v, d), \alpha_m \rangle ds \\ & + \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle -B(v, d) + B(\tilde{v}_m, \tilde{d}_m), \alpha_m \rangle ds = 0, \\ & \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle \beta_4^b - B(\tilde{d}_m, \tilde{v}_m), \alpha_m \rangle ds = \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle \beta_4^b - B(d, v), \alpha_m \rangle ds \\ & + \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle B(d, v) - B(\tilde{d}_m, \tilde{v}_m), \alpha_m \rangle ds = 0, \end{aligned}$$

and

$$\begin{aligned} & \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle -\beta_1^b + B_1(\tilde{v}_m, \tilde{\Psi}_m), A_\gamma \rho_m \rangle ds = \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle -\beta_1^b + B_1(v, \Psi), A_\gamma \rho_m \rangle ds \\ & + \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle -B_1(v, \Psi) + B_1(\tilde{v}_m, \tilde{\Psi}_m), A_\gamma \rho_m \rangle ds = 0. \end{aligned}$$

(v). We must now prove that

$$\lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle h_1^b(s) - f_1(s, \tilde{v}_m), \theta_m \rangle ds = 0.$$

Using (104) and the fact that

$$\begin{aligned} & 1_{[0, \tau_n]} \sigma(t) (h_1^b(t) - f_1(t, v)) \in M_{\mathcal{F}_t}^2(0, T, V_1^*) \quad \text{and} \\ & 1_{[0, \tau_n]} \sigma(t) (f_1(t, v) - f_1(t, \tilde{v}_m)) \rightarrow 0 \quad \text{in } M_{\mathcal{F}_t}^2(0, T, V_1^*) \quad \text{as } m \rightarrow \infty, \end{aligned}$$

we deduce that

$$\begin{aligned} & \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle h_1^b(s) - f_1(s, v), \theta_m \rangle ds = 0, \\ & \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle f_1(s, v) - f_1(s, \tilde{v}_m), \theta_m \rangle ds = 0. \end{aligned} \quad (114)$$

Furthermore, we deduce that

$$\begin{aligned} & \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle h_1^b(s) - f_1(s, \tilde{v}_m), \theta_m \rangle ds = \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle h_1^b(s) - f_1(s, v), \theta_m \rangle ds \\ & + \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle f_1(s, v) - f_1(s, \tilde{v}_m), \theta_m \rangle ds = 0. \end{aligned} \quad (115)$$

Finally we can conclude from (i)-(v) that the right side of (100) goes to 0 as m goes to $+\infty$.

Now using the fact that $1_{[0, \tau_n]} \sigma(t) \leq 1$, we deduce from (100) that

$$\begin{aligned} & \lim_{m \rightarrow \infty} \mathbb{E} |(\theta_m, \beta_m, \psi_m)(\tau_n)|_{\mathbb{Y}}^2 = \lim_{m \rightarrow \infty} \mathbb{E} \left(|(\tilde{v}_m, \tilde{d}_m, \tilde{\Psi}_m)(\tau_n) - (v_m, d_m, \Psi_m)(\tau_n)|_{\mathbb{Y}}^2 \right) = 0, \\ & \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} K_2(s) ds = \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} (|\theta_m|^2 + |\alpha_m|^2 + |A_\gamma \rho_m|_{L^2}^2) ds \\ & = \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} (|\tilde{v}_m - v_m|^2 + |\tilde{d}_m - d_m|^2 + |A_\gamma(\tilde{\Psi}_m - \Psi_m)|_{L^2}^2) ds = 0, \\ & \mathbb{E} \int_0^{\tau_n} \|h_2^b - f_2(s, v)\|_{\mathcal{L}^2(\mathbb{K}, H_1)}^2 ds = 0. \end{aligned} \quad (116)$$

Claim 2. The following hold true.

$$\begin{aligned}
f_2(t, v) &= h_2^b \quad \text{in} \quad M_{\mathcal{F}_t}^2(0, T, \mathcal{L}^2(\mathbb{K}, H_1)), \\
B(v, v) &= \beta_0^b \quad \text{in} \quad M_{\mathcal{F}_t}^2(0, T, V_1^*), \\
B(d, d) &= \beta_2^b \quad \text{in} \quad M_{\mathcal{F}_t}^2(0, T, V_1^*), \\
R_0(A_\gamma \Psi, \Psi) &= r_0^b \quad \text{in} \quad M_{\mathcal{F}_t}^2(0, T, V_1^*), \\
B(v, d) &= \beta_3^b \quad \text{in} \quad M_{\mathcal{F}_t}^2(0, T, V_2^*), \\
B(d, v) &= \beta_4^b \quad \text{in} \quad M_{\mathcal{F}_t}^2(0, T, V_2^*), \\
B_1(v, \Psi) &= \beta_1^b \quad \text{in} \quad M_{\mathcal{F}_t}^2(0, T, H_3), \\
f_1(t, v) &= h_1^b(t) \quad \text{in} \quad M_{\mathcal{F}_t}^2(0, T, V_1^*), \\
f_\gamma(\Psi) &= f^b \quad \text{in} \quad M_{\mathcal{F}_t}^2(0, T, H_3).
\end{aligned} \tag{117}$$

To prove (117)₁, we can note that from (116)₃ and the fact that the sequence $\{\tau_n; n \geq 1\}$ is increasing to T , we can deduce that

$$h_2^b = f_2(t, v) \quad \text{in} \quad M_{\mathcal{F}_t}^2(0, T, \mathcal{L}^2(\mathbb{K}, H_1)). \tag{118}$$

This proves (117)₁.

To prove (117)₂, we proceed as follows. From (116)₂ and (83), we also write

$$(v_m, d_m, \Psi_m)|_{[0, \tau_n]} \rightarrow (v, d, \Psi)|_{[0, \tau_n]} \quad \text{in} \quad M_{\mathcal{F}_t}^2(0, T, \mathbb{V}).$$

Furthermore, for all $w \in M_{\mathcal{F}_t}^\infty(0, T, V_1)$, we have

$$\mathbb{E} \int_0^{\tau_n} \langle B(v, v) - B(v_m, v_m), w \rangle ds \leq c \|w\|_{M_{\mathcal{F}_t}^\infty(0, T, V_1)} \times \mathbb{E} \int_0^{\tau_n} \|v_m - v\|^{1/2} |v_m - v|_{L^2}^{1/2} (\|v\| + \|v_m\|) ds,$$

which gives

$$\lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \langle B(v, v) - B(v_m, v_m), w \rangle ds = 0. \tag{119}$$

Focusing (81)₃ and (119), we deduce that

$$\lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \langle B(v, v) - \beta_0^b, w \rangle ds = 0, \quad \forall w \in M_{\mathcal{F}_t}^\infty(0, T, V_1).$$

Since $\tau_n \uparrow T$ and $M_{\mathcal{F}_t}^\infty(0, T, V_1)$ is dense in $M_{\mathcal{F}_t}^2(0, T, V_1)$, we can conclude that

$$B(v, v) = \beta_0^b \quad \text{in} \quad M_{\mathcal{F}_t}^2(0, T, V_1^*),$$

this proves (117)₂.

Using the same methods as in (117)₂, (81)₃ and (119), we also prove

$$B(d, d) = \beta_2^b \quad \text{in} \quad M_{\mathcal{F}_t}^2(0, T, V_1^*),$$

this proves (117)₃.

To prove (117)₄, we can note that

$$\begin{aligned}
&\mathbb{E} \int_0^{\tau_n} \langle R_0(A_\gamma \Psi, \Psi) - R_0(A_\gamma \Psi_m, \Psi_m), w \rangle ds \leq c \|w\|_{M_{\mathcal{F}_t}^\infty(0, T, V_1)} \times \\
&\mathbb{E} \int_0^{\tau_n} (\|\Psi\|^{1/2} |A_\gamma(\Psi_m - \Psi)|_{L^2}^{1/2} |A_\gamma \Psi|_{L^2}^{1/2} + \|\Psi_m - \Psi\|^{1/2} |A_\gamma(\Psi_m - \Psi)|_{L^2}^{1/2} |A_\gamma \Psi_m|_{L^2}) ds,
\end{aligned}$$

which gives

$$\lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \langle R_0(A_\gamma \Psi, \Psi) - R_0(A_\gamma \Psi_m, \Psi_m), w \rangle ds = 0.$$

From (81)₄ and (3), we deduce that

$$\mathbb{E} \int_0^{\tau_n} \langle R_0(A_\gamma \Psi, \Psi) - r_0^b, w \rangle ds = 0, \quad \forall w \in M_{\mathcal{F}_t}^\infty(0, T, V_1),$$

which gives

$$R_0(A_\gamma \Psi, \Psi) = r_0^b \quad \text{in} \quad M_{\mathcal{F}_t}^2(0, T, V_1^*),$$

and (117)₄ is proved.

Using the same technique, one easily proves the other equalities in (117). We derive from Claims 1 and 2 that (v, d, Ψ) is the solution to (16).

All in all, let's note that the estimate (62) is a derivation as in (75) that uses the stopping time given by (71). This concludes the proof of Theorem 3.1.

The following Proposition is a proof that the Galerkin solution (v_m, d_m, Ψ_m) converges to the unique solution (v, d, Ψ) of problem (16).

Proposition 3.3. *Let (v_m, d_m, Ψ_m) the solution of (63) and (v, d, Ψ) the unique solution of (16). We get*

$$\begin{aligned} \lim_{m \rightarrow \infty} \mathbb{E} |(v_m, d_m, \Psi_m) - (v, d, \Psi)|_{\mathbb{V}}^2 &= 0, \quad \forall t \in [0, T], \\ \lim_{m \rightarrow \infty} \mathbb{E} |(v_m, d_m, \Psi_m)(s) - (v, d, \Psi)(s)|_{\mathbb{V}}^2 &= 0, \quad \forall t \in [0, T]. \end{aligned}$$

To justify the results obtained above, we recall the following lemma that will be applied.

Lemma 3.1. [9, 15] *Consider $\{Q_m; m \geq 1\} \subset M_{\mathcal{F}_t}^2(0, T, \mathbb{R})$ as a sequence of continuous real processes, and $\{\tau_n; n \geq 1\}$ as a sequence of $\mathcal{F}_t$ -stopping times verifying $\tau_n \uparrow T$; $\sup_{m \geq 1} \mathbb{E} |Q_m(T)|^2 < \infty$, and $\lim_{m \rightarrow \infty} \mathbb{E} |Q_m(\tau_n)| = 0$, for all $n \geq 1$. Then $\lim_{m \rightarrow \infty} \mathbb{E} |Q_m(T)| = 0$.*

Proof of Proposition 3.3.

Considering the lemma 3.1 and applying it to $Q_m(t) = |(v, d, \Psi) - (v_m, d_m, \Psi_m)|_{\mathbb{V}}^2$, $\sigma_n = \tau_n$ and focusing on (62), (75), (116)₁ and the uniqueness of (v, d, Ψ) , we do deduce that the set of sequences highlighted by (63) verifies

$$\lim_{m \rightarrow \infty} \mathbb{E} |(v, d, \Psi) - (v_m, d_m, \Psi_m)|_{\mathbb{V}}^2 = 0, \quad \forall t \in [0, T]. \quad (120)$$

In the same vein, using the lemma 3.1 for $Q_m(t) = \int_0^t \|(v, d, \Psi)(s) - (v_m, d_m, \Psi_m)(s)\|_{\mathbb{V}}^2 ds$ and taking into account (62), (116)₂, it is deduced that the sequence (v_m, d_m, Ψ_m) tends strongly towards (v, d, Ψ) in $M_{\mathcal{F}_t}^2(0, T, \mathbb{V})$, i.e.,

$$\lim_{m \rightarrow \infty} \mathbb{E} \int_0^t \|(v, d, \Psi)(s) - (v_m, d_m, \Psi_m)(s)\|_{\mathbb{V}}^2 ds = 0, \quad \forall t \in [0, T]. \quad (121)$$

This achieves the proof of Proposition 3.3.

4 Regularity of the solution for the stochastic 2D AC-MHD model

Here, we prove a regularity result of the unique probabilistic strong solution of the stochastic 2D Allen-Cahn-Magnetohydrodynamics model in a higher space. For this, we need the following assumptions on f_1 and f_2 .

Assumption 2. Suppose that $\nabla f_1 : \Omega \times (0, T) \times H_1 \longrightarrow V_1^*$, $\nabla f_2 : \Omega \times (0, T) \times H_1 \longrightarrow \mathcal{L}^2(\mathbb{R}; H_1)$, where $\nabla f_1, \nabla f_2$ are measurable functions that respect the Lipschitz conditions, for all $v_1, v_2 \in V_1$.

$$\begin{aligned} \|\nabla f_1(t, v_1) - \nabla f_1(t, v_2)\|_{V_1^*} &\leq L_1 \|\nabla v_1 - \nabla v_2\|_{L^2}, \\ \nabla f_1(t, 0) &\in M_{\mathcal{F}_t}^2(0, T; V_1^*), \\ \|\nabla f_2(t, v_1) - \nabla f_2(t, v_2)\|_{\mathcal{L}^2(\mathbb{R}; H_1)} &\leq L_2 \|\nabla v_1 - \nabla v_2\|_{L^2}, \\ \nabla f_2(t, 0) &\in M_{\mathcal{F}_t}^2(0, T; \mathcal{L}^2(\mathbb{R}; H_1)). \end{aligned} \quad (122)$$

In the following theorem, we prove the regularity of strong solutions to the stochastic Allen-Cahn-Magnetohydrodynamic model in a higher space.

Theorem 4.1. Assume that **Assumption 1** - **Assumption 2** hold and $(v_0, d_0, \Psi_0) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{V})$ such that $\mathbb{E}[\mathcal{E}(v_0, d_0, \Psi_0)]^2 < +\infty$. Then, the unique strong solution (v, d, Ψ) of problem (16) satisfies the following regularity

$$(v, d, \Psi) \in L^2(\Omega, C(0, T; \mathbb{V})) \cap L^2(\Omega, (0, T; G^{1/2}(\mathbb{V})))$$

For all $T > 0$, $\exists N$, the following estimate is true:

$$\begin{aligned} \mathbb{E} \sup_{[0, T]} (\|v\|^2 + \|d\|^2 + |\Delta \Psi|_{L^2}^2 + \|\mu\|^2) &+ \mathbb{E} \left(\int_0^T (|\Delta v|_{L^2}^2 + |\Delta d|_{L^2}^2 + |\Psi_t|_{L^2}^2 + |\mu|_{L^2}^2) dt \right) \\ &\leq ce^{2N} \mathbb{E} (\|v_0\|^2 + \|d_0\|^2 + |\Delta \Psi_0|_{L^2}^2 + |\mu_0|_{L^2}^2) + c \mathbb{E} \left(\int_0^T \|\nabla f_1(t, 0)\|_{V_1^*}^2 dt \right) \\ &+ c \mathbb{E} \left(\int_0^T \|\nabla f_2(t, 0)\|_{\mathcal{L}^2(\mathbb{R}; H_1)}^2 dt \right). \end{aligned} \quad (123)$$

Proof : By applying the Itô formula to the first equation of the system (16), we have

$$\begin{aligned} \|\nabla v\|_{L^2}^2 + 2 \int_0^t |\Delta v|_{L^2}^2 ds &= \|\nabla v_0\|_{L^2}^2 - 2 \int_0^t \langle B(v, v), \Delta v \rangle ds + 2 \int_0^t \langle B(d, d), \Delta v \rangle ds \\ + 2 \int_0^t \langle R_0(\mu, \Psi), \Delta v \rangle ds &+ 2 \int_0^t \langle \nabla f_1(s, v), \nabla v \rangle ds + 2 \int_0^t \langle \nabla f_2(s, v), \nabla v \rangle dW_s + \int_0^t \|\nabla f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{R}; H_1)}^2 ds. \end{aligned} \quad (124)$$

Thereafter, we carry out the scalar product with Δd on the second equation of the system (16), and integrate the resulting equality from 0 to t get

$$\|\nabla d\|_{L^2}^2 + 2 \int_0^t |\Delta d|_{L^2}^2 ds = \|\nabla d_0\|_{L^2}^2 - 2 \int_0^t \langle B(d, v), \Delta d \rangle ds + 2 \int_0^t \langle B(v, d), \Delta d \rangle ds. \quad (125)$$

Similarly, multiplying (16)₃ by $2\Psi_t - 2\mu$, after some calculation, we get

$$\langle \Psi_t + \mu, 2\Psi_t - 2\mu \rangle + 2\langle B_1(v, \Psi), \Psi_t - \mu \rangle = 0, \quad (126)$$

which leads to

$$2\langle \Psi_t, \Psi_t \rangle - 2\langle \Psi_t, \mu \rangle + 2\langle \mu, \Psi_t \rangle - 2\langle \mu, \mu \rangle + 2\langle B_1(v, \Psi), \Psi_t - \mu \rangle = 0. \quad (127)$$

Then

$$2\|\Psi_t\|_{L^2}^2 + 2\|\mu\|_{L^2}^2 + 2\langle B_1(v, \Psi), \Psi_t - \mu \rangle = 4\|\mu\|_{L^2}^2. \quad (128)$$

Applying the scalar product with $2\Delta \Psi_t$ in the fourth equation of (16), we have

$$-2\langle \nabla \mu, \nabla \Psi_t \rangle = \langle A_\gamma \Psi, 2\Delta \Psi_t \rangle + 2\langle f_\gamma(\Psi), \Delta \Psi_t \rangle. \quad (129)$$

It is clear that

$$-2\langle \nabla \mu, \nabla \Psi_t \rangle = \langle -\Delta \Psi + \gamma \Psi, 2\Delta \Psi_t \rangle - 2\langle f'_\gamma(\Psi) \nabla \Psi, \nabla \Psi_t \rangle. \quad (130)$$

Thus

$$2\langle \nabla \mu, \nabla \Psi_t \rangle + 2\gamma \langle \Psi, \Delta \Psi_t \rangle - \frac{d}{dt} |\Delta \Psi|_{L^2}^2 = 2\langle f'_\gamma(\Psi) \nabla \Psi, \nabla \Psi_t \rangle. \quad (131)$$

Differentiating (16)₄ with respect to time, we get

$$\mu_t = A_\gamma \Psi_t + f'_\gamma(\Psi) \Psi_t. \quad (132)$$

After some algebra, we take the scalar product in (132) by 2μ to have

$$\langle \mu_t, 2\mu \rangle = 2\langle -\Delta \Psi_t + \gamma \Psi_t, \mu \rangle + 2\langle f'_\gamma(\Psi) \Psi_t, \mu \rangle. \quad (133)$$

We derive that

$$\frac{d|\mu|_{L^2}^2}{dt} = 2\langle \nabla \mu, \nabla \Psi_t \rangle + 2\gamma \langle \Psi_t, \mu \rangle + 2\langle f'_\gamma(\Psi) \Psi_t, \mu \rangle. \quad (134)$$

Adding (131) and (134), we have

$$\frac{d}{dt} (|\mu|_{L^2}^2 + |\Delta \Psi|_{L^2}^2 - 2|\Delta \Psi|_{L^2}^2 + \gamma^2 |\Psi|_{L^2}^2) = 2\langle f'_\gamma(\Psi) \nabla \Psi, \nabla \Psi_t \rangle + 2\langle f'_\gamma(\Psi) \Psi_t, \mu \rangle, \quad (135)$$

since

$$2\gamma \langle \Psi, \Delta \Psi_t \rangle - 2\gamma \langle \Psi_t, \mu \rangle = 2\gamma \langle \Psi, \Delta \Psi_t \rangle + 2\gamma \langle \Psi, \mu_t \rangle = 2\gamma \langle \Psi, \Delta \Psi_t \rangle + 2\gamma \langle \Psi, -\Delta \Psi_t \rangle + 2\gamma \langle \Psi, \gamma \Psi_t \rangle = \gamma^2 \frac{d}{dt} |\Psi|_{L^2}^2. \quad (136)$$

Adding the results of the equality (124), (125), (135), and integrating (128) and (131) from 0 to t , we have

$$\begin{aligned} & |\nabla v|_{L^2}^2 + |\Delta \Psi|_{L^2}^2 + |\nabla d|_{L^2}^2 + |\mu|_{L^2}^2 + 2 \int_0^t (|\Delta v|_{L^2}^2 + |\mu|_{L^2}^2 + |\Delta d|_{L^2}^2 + |\Psi_t|_{L^2}^2) ds = |\nabla v_0|_{L^2}^2 + |\Delta \Psi_0|_{L^2}^2 \\ & + |\nabla d_0|_{L^2}^2 + \|\mu_0\|^2 - \gamma^2 |\Psi|_{L^2}^2 + 2|\Delta \Psi|_{L^2}^2 + 4|\mu|_{L^2}^2 - 2 \int_0^t \langle B(v, v), \Delta v \rangle ds + 2 \int_0^t \langle R_0(\mu, \Psi), \Delta v \rangle ds \\ & + 2 \int_0^t \langle \nabla f_1(s, v), \nabla v \rangle ds + 2 \int_0^t \langle \nabla f_2(s, v), \nabla v \rangle dW_s + \int_0^t \|\nabla f_2(s, v)\|_{\mathcal{L}^2(\mathbb{R}; H_1)}^2 ds - 2 \int_0^t \langle B_1(v, \Psi), \Psi_s - \mu \rangle ds \\ & - 2 \int_0^t \langle f'_\gamma(\Psi) \nabla \Psi, \nabla \Psi_s \rangle ds + 2 \int_0^t \langle f'_\gamma(\Psi) \Psi_s, \mu \rangle ds + 2 \int_0^t \langle B(d, d), \Delta v \rangle ds + 2 \int_0^t \langle B(v, d), \Delta d \rangle ds \\ & - 2 \int_0^t \langle B(d, v), \Delta d \rangle ds. \end{aligned} \quad (137)$$

As in [28, 40], we recall the following estimates:

$$\begin{aligned} |\langle B(v, v), \Delta v \rangle| & \leq |v|_{L^4} |\nabla v|_{L^4} |\Delta v|_{L^2} \leq \frac{1}{8} |\Delta v|_{L^2}^2 + c|v|_{L^2}^2 \|v\|^4 \\ & \leq \frac{1}{8} |\Delta v|_{L^2}^2 + c|v|_{L^2}^2 \|v\|^2 (\|v\|^2 + \|d\|^2 + |\Delta \Psi|_{L^2}^2 + |\mu|_{L^2}^2), \end{aligned} \quad (138)$$

$$\begin{aligned} |\langle B(d, d), \Delta v \rangle| & \leq |d|_{L^4} |\nabla d|_{L^4} |\Delta v|_{L^2} \leq \frac{1}{8} |\Delta v|_{L^2}^2 + c|d|_{L^2}^2 \|d\|^4 \\ & \leq \frac{1}{8} |\Delta v|_{L^2}^2 + c|d|_{L^2}^2 \|d\|^2 (\|v\|^2 + \|d\|^2 + |\Delta \Psi|_{L^2}^2 + |\mu|_{L^2}^2), \end{aligned} \quad (139)$$

$$\begin{aligned} |\langle B(d, v) - B(v, d), \Delta d \rangle| & \leq \frac{1}{8} |\Delta d|_{L^2}^2 + c|d|_{L^2}^2 \|v\|^4 + \frac{1}{8} |\Delta d|_{L^2}^2 + c|v|_{L^2}^2 \|d\|^4 \\ & \leq \frac{1}{4} |\Delta d|_{L^2}^2 + c(|d|_{L^2}^2 \|v\|^4 + |v|_{L^2}^2 \|d\|^4) (\|v\|^2 + \|d\|^2 + |\Delta \Psi|_{L^2}^2 + |\mu|_{L^2}^2), \end{aligned} \quad (140)$$

$$\begin{aligned} |\langle B_1(v, \Psi), \Psi_t - \mu \rangle| & \leq |v|_{L^4} |\nabla \Psi|_{L^4} |\Psi_t - \mu|_{L^2} \\ & \leq \frac{1}{4} |\Psi_t|_{L^2}^2 + \frac{1}{2} |\Delta \mu|_{L^2}^2 + c|v|_{L^2}^2 \|v\|^2 \\ & + c\|\Psi\|^2 (\|v\|^2 + \|d\|^2 + |\Delta \Psi|_{L^2}^2 + |\mu|_{L^2}^2), \end{aligned} \quad (141)$$

$$\begin{aligned}
| \langle R_0(\mu, \Psi), \Delta v \rangle | &\leq |\Delta v|_{L^2} |\nabla \Psi|_{L^4} |\mu|_{L^4} \\
&\leq \frac{1}{8} |\Delta v|_{L^2}^2 + c(|\mu|^2 + \|\Psi\|^2)(\|v\|^2 + \|d\|^2 + |\Delta \Psi|_{L^2}^2 + |\mu|_{L^2}^2),
\end{aligned} \tag{142}$$

$$\begin{aligned}
| \langle f'_\gamma(\Psi) \nabla \Psi, \nabla \Psi_t \rangle | &= | \langle f''_\gamma(\Psi) (\nabla \Psi)^2 + f'_\gamma(\Psi) \Delta \Psi, \Psi_t \rangle | \\
&= | \langle f''(\Psi) (\nabla \Psi)^2 + (f'(\Psi) - \alpha^{-1} \nu_2 \gamma) \Delta \Psi, \Psi_t \rangle | \\
&\leq \frac{1}{8} |\Psi_t|_{L^2}^2 + c |\Psi|_{L^2}^2 \|\Psi\|^4 + \frac{1}{8\gamma} |\Delta \Psi|_{L^2}^2 + | \langle \alpha^{-1} \nu_2 \gamma \Delta \Psi, \Psi_t \rangle | \\
&\leq \frac{1}{8} |\Psi_t|_{L^2}^2 + c |\Psi|_{L^2}^2 \|\Psi\|^4 + c(\|v\|^2 + \|d\|^2 + |\Delta \Psi|_{L^2}^2 + |\mu|_{L^2}^2),
\end{aligned} \tag{143}$$

$$\begin{aligned}
\|\nabla f_2(s, v)\|_{\mathcal{L}^2(\mathbb{R}; H_1)}^2 &\leq 2cL_2^2 \|\nabla v\|_{L^2}^2 + c \|\nabla f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{R}; H_1)}^2 \\
&\leq 2cL_2^2 (\|v\|^2 + \|d\|^2 + |\Delta \Psi|_{L^2}^2 + |\mu|_{L^2}^2) + c \|\nabla f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{R}; H_1)}^2,
\end{aligned} \tag{144}$$

$$\begin{aligned}
\langle \nabla f_1(s, v), \nabla v \rangle &\leq L_1 \|v\| |\Delta v| + c \|\nabla f_1(s, 0)\|_{V_1^*} |\Delta v| \\
&\leq \frac{1}{4} |\Delta v|_{L^2}^2 + cL_1^2 \|v\|^2 + c \|\nabla f_1(s, 0)\|_{V_1^*}^2 \\
&\leq \frac{1}{4} |\Delta v|_{L^2}^2 + cL_1^2 (\|v\|^2 + \|d\|^2 + |\Delta \Psi|_{L^2}^2 + |\mu|_{L^2}^2) + c \|\nabla f_1(s, 0)\|_{V_1^*}^2,
\end{aligned} \tag{145}$$

$$\begin{aligned}
| \langle f'_\gamma(\Psi) \Psi_t, \mu \rangle | &= | \langle (f'(\Psi) - \alpha^{-1} \nu_2 \gamma) \Psi_t, \mu \rangle | \\
&\leq | \langle f'(\Psi) \Psi_t, \mu \rangle | + | \langle \alpha^{-1} \nu_2 \gamma \Psi_t, \mu \rangle | \\
&\leq | \langle (3\Psi^2 - 1) \Psi_t, \mu \rangle | + \alpha^{-1} \nu_2 \gamma |\Psi_t|_{L^2} \|\mu\|_{L^2} \\
&\leq 3|\Psi^2 - 1|_{L^2} |\Psi_t|_{L^\infty} + 2|\Psi_t|_{L^2} \|\mu\|_{L^2} + \alpha^{-1} \nu_2 \gamma |\Psi_t|_{L^2} \|\mu\|_{L^2} \\
&\leq \frac{1}{8} |\Psi_t|_{L^2}^2 + c(\|v\|^2 + \|d\|^2 + |\Delta \Psi|_{L^2}^2 + |\mu|_{L^2}^2).
\end{aligned} \tag{146}$$

From the Theorem 3.1, we can define a stopping time τ_N by

$$\tau_N = \inf \left\{ t \leq T; \int_0^t (|v|^2 \|v\|^2 + |d|^2 \|d\|^2 + |d|^2 \|v\|^4 + |v|^2 \|d\|^4 + \|\mu\|^2 + 2\|\Psi\|^2 + L_1^2 + L_2^2 + 2) ds \geq N \right\}. \tag{147}$$

Let

$$\begin{aligned}
K(t) &= c(|v|^2 \|v\|^2 + |d|^2 \|d\|^2 + |d|^2 \|v\|^4 + |v|^2 \|d\|^4 + \|\mu\|^2 + 2\|\Psi\|^2 + L_1^2 + L_2^2 + 2), \\
Y(t) &= \|v\|^2 + \|d\|^2 + |\Delta \Psi|_{L^2}^2 + |\mu|_{L^2}^2, \\
\sigma(t) &= \exp \left(-2 \int_0^t K(s) ds \right).
\end{aligned} \tag{148}$$

In view of (137) and from (138)-(148), we have the following inequalities

$$\begin{aligned}
\sigma(t) dY(t) + Y(t) d\sigma(t) + 2\sigma(t) (|\Delta v|_{L^2}^2 + |\mu|_{L^2}^2 + |\Delta d|_{L^2}^2 + |\Psi_t|_{L^2}^2) dt &\leq 2\sigma(t) \left[\frac{1}{8} |\Delta v|_{L^2}^2 + \frac{1}{8} |\Delta v|_{L^2}^2 \right. \\
&+ \frac{1}{4} |\Delta v|_{L^2}^2 + c \|\nabla f_1(s, 0)\|_{V_1^*}^2 + c \|\nabla f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{R}; H_1)}^2 + \frac{1}{8} |\Psi_t|_{L^2}^2 + c |\Psi|_{L^2}^2 \|\Psi\|^4 + \frac{1}{4} |\Psi_t|_{L^2}^2 \\
&+ \frac{1}{2} |\Delta \mu|_{L^2}^2 + c |v|_{L^2}^2 \|v\|^2 + \frac{1}{8} |\Psi_t|_{L^2}^2 + \frac{1}{8} |\Delta v|_{L^2}^2 + \frac{1}{4} |\Delta d|_{L^2}^2 \Big] dt + 2\sigma(t) \langle \nabla f_2(t, v), \nabla v \rangle dW_t.
\end{aligned} \tag{149}$$

Consequently

$$\begin{aligned}
d(\sigma(t)Y(t)) + \sigma(t) (|\Delta v|_{L^2}^2 + |\mu|_{L^2}^2 + |\Delta d|_{L^2}^2 + |\Psi_t|_{L^2}^2) dt &\leq 2\sigma(t) \left[\frac{1}{8} |\Delta v|_{L^2}^2 + c |\Psi|_{L^2}^2 \|\Psi\|^4 \right. \\
&+ c |v|_{L^2}^2 \|v\|^2 + \|\nabla f_2(t, 0)\|_{\mathcal{L}^2(\mathbb{R}; H_1)}^2 \Big] dt + 2c\sigma(t) \|\nabla f_1(t, 0)\|_{V_1^*}^2 + 2\sigma(t) \langle \nabla f_2(t, v), \nabla v \rangle dW_t.
\end{aligned} \tag{150}$$

Thus

$$\begin{aligned}
&\sigma(t)Y(t) + \int_0^{t \wedge \tau_N} \sigma(s) (|\Delta v|_{L^2}^2 + |\mu|_{L^2}^2 + |\Delta d|_{L^2}^2 + |\Psi_t|_{L^2}^2) ds \leq \sigma(0)Y(0) \\
&+ 2c \int_0^{t \wedge \tau_N} \sigma(s) \left[\|\nabla f_1(s, 0)\|_{V_1^*}^2 + \|\nabla f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{R}; H_1)}^2 \right] ds + 2 \int_0^{t \wedge \tau_N} \sigma(t) \langle \nabla f_2(s, v), \nabla v \rangle dW_s.
\end{aligned} \tag{151}$$

Applying the expectation and the Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned}
\mathbb{E} \int_0^{t \wedge \tau_N} \sigma(t) \langle \nabla f_2(s, v), \nabla v \rangle dW_s &\leq c \mathbb{E} \sup_{s \in [0, t \wedge \tau_N]} |\nabla v|_{L^2}^2 \int_0^{t \wedge \tau_N} \|\nabla f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{R}; H_1)}^2 ds \\
&\leq c L_2^2 \mathbb{E} \sup_{s \in [0, t \wedge \tau_N]} |\nabla v|_{L^2}^2 + c \mathbb{E} \sup_{s \in [0, t \wedge \tau_N]} \int_0^{t \wedge \tau_N} \|\nabla f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{R}; H_1)}^2 ds \\
&\leq c L_2^2 \mathbb{E} \sup_{s \in [0, t \wedge \tau_N]} Y(s) + c \mathbb{E} \sup_{s \in [0, t \wedge \tau_N]} \int_0^{t \wedge \tau_N} \|\nabla f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{R}; H_1)}^2 ds.
\end{aligned}$$

Ultimately

$$\begin{aligned}
\mathbb{E} \sup_{s \in [0, t \wedge \tau_N]} \sigma(s) Y(s) + \mathbb{E} \int_0^{t \wedge \tau_N} \sigma(s) (|\Delta v|_{L^2}^2 + |\mu|_{L^2}^2 + |\Delta d|_{L^2}^2 + |\Psi_t|_{L^2}^2) ds &\leq \mathbb{E} \sigma(0) Y(0) \\
+ 2c \mathbb{E} \int_0^{t \wedge \tau_N} \sigma(s) \left[\|\nabla f_1(s, 0)\|_{V_1^*}^2 + \|\nabla f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{R}; H_1)}^2 \right] ds. & \quad (152)
\end{aligned}$$

Let N be a large quantity and T be a fixed time. Using the definition of τ_N , we get $T \wedge \tau_N = T$

$$\begin{aligned}
\mathbb{E} \sup_{s \in [0, T]} Y(s) + \mathbb{E} \int_0^T \left[|\Delta v|_{L^2}^2 + |\Delta d|_{L^2}^2 + |\Psi_t|_{L^2}^2 + |\mu|_{L^2}^2 \right] ds &\leq c e^{2N} \mathbb{E} Y(0) \\
+ 2c \mathbb{E} \int_0^T \left[\|\nabla f_1(s, 0)\|_{V_1^*}^2 + \|\nabla f_2(s, 0)\|_{\mathcal{L}^2(\mathbb{R}; H_1)}^2 \right] ds. & \quad (153)
\end{aligned}$$

This achieves the proof of Theorem 4.1.

Remark 4.1. Theorem 3.1 justified that $(v, d, \Psi) \in L^4(\Omega, C(0, T; \mathbb{Y})) \cap L^4(\Omega, (0, T; \mathbb{V}))$, and $\mu \in L^2(\Omega, (0, T; D(B^{1/2})))$.

In this paper, we have the following regularity

$$(v, d, \Psi) \in L^2(\Omega, C(0, T; \mathbb{V})) \cap L^2(\Omega, (0, T; B^{1/2}(\mathbb{V})),$$

and $\mu \in L^2(\Omega, C(0, T; H_1)) \cap L^2(\Omega, (0, T; D(B)))$, $\Psi_t \in L^2(\Omega, (0, T; H_1))$.

Remark 4.2. From Theorem 4.1, we admit that the solution (v, d, Ψ) of system (2) belongs to $C([0, T], \mathbb{V})$ for the initial value $(v_0, d_0, \Psi_0) \in \mathbb{V}$ and from [53], we deduce that there exists a unique weak solution in $C(0, T; \mathbb{Y}) \cap L^2(0, T; \mathbb{V})$ for the initial value $(v_0, d_0, \Psi_0) \in \mathbb{Y}$, $(v, d, \Psi) \in \mathbb{V}$.

Due the fact that the solution is unique and $\mathbb{V} \subset \mathbb{Y}$ is compact, we conclude that for the initial value $(v_0, d_0, \Psi_0) \in \mathbb{Y}$, $(v, d, \Psi) \in \mathbb{V}$. This is because the standard Kryloff-Bogoliubov procedure yields the existence of an invariant measure on $\mathbb{Y}$, loaded on $\mathbb{V}$.

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